OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS HIGHER LEVEL:

STATISTICS

COURSE COMPANION

Josip Harcet
Lorraine Heinrichs
Palmira Mariz Seiler
Marlene Torres Skoumal

OXFORD

1 Exploring further probability distributions

Skills check

1 a Mode $(X) = -1, 1$ because those two values share the highest probability of 0.3.

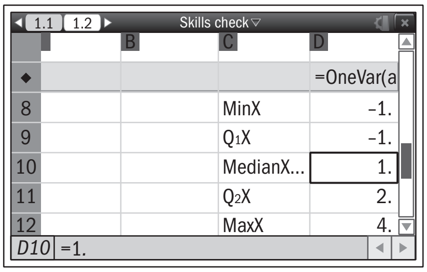
Median, $m = 1$ since $P(X \le 0) = 0.4$ and $P(X \le 1) = 0.7$

$$\mu = E(X) = \sum_{i=1}^{6} x_i p_i = -1 \times 0.3 + 0 \times 0.1 + 1 \times 0.3$$
$$+ 2 \times 0.1 + 3 \times 0.05 + 4 \times 0.15 = 0.95$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\sum_{i=1}^{6} x_i^2 p_i - \mu^2}$$

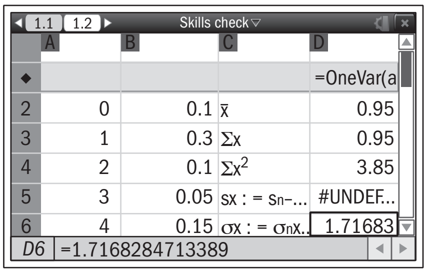
$$\sum_{i=1}^{6} x_i^2 P_i = (-1)^2 \times 0.3 + 0^2 \times 0.1 + 1^2 \times 0.3$$
$$+ 2^2 \times 0.1 + 3^2 \times 0.05 + 4^2 \times 0.15$$
$$= 3.85$$

$$\sigma = \sqrt{3.85 - 0.9025} = 1.72$$



◆			=OneVar(a
8		MinX	-1.
9		Q₁X	-1.
10		MedianX...	1.
11		Q₂X	2.
12		MaxX	4.

D10 =1.



	A	B	C	D
◆				=OneVar(a
2	0	0.1	x̄	0.95
3	1	0.3	Σx	0.95
4	2	0.1	Σx²	3.85
5	3	0.05	sx := sn-...	#UNDEF...
6	4	0.15	σx := σnx...	1.71683

D6 =1.7168284713389

b

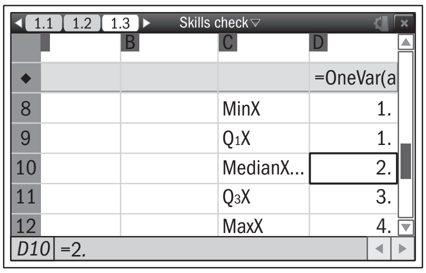
x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

Mode $(X) = 1$ because it has the highest probability (0.4)

Median, $m = 2$ since $P(X \le 1) = 0.4$ and $P(X \le 2) = 0.7$

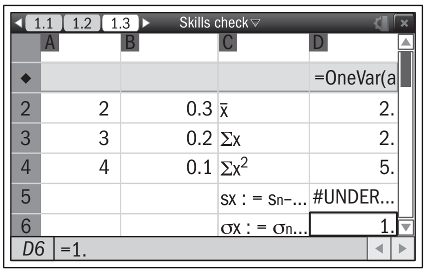
$$\mu = E(X) = \sum_{i=1}^{4} x_i p_i = 1 \times 0.4 + 2 \times 0.3$$
$$+ 3 \times 0.2 + 4 \times 0.1 = 2$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\sum_{i=1}^{4} x_i^2 p_i - \mu^2}$$
$$= \sqrt{1^2 \times 0.4 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.1 - 2^2}$$
$$= \sqrt{5-4} = 1$$



◆			=OneVar(a
8		MinX	1.
9		Q₁X	1.
10		MedianX...	2.
11		Q₃X	3.
12		MaxX	4.

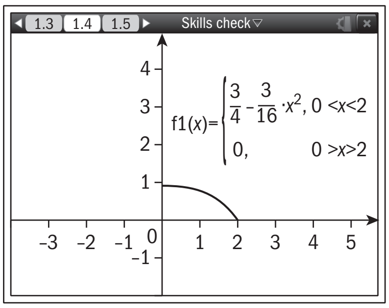
D10 =2.



	A	B	C	D
◆				=OneVar(a
2	2	0.3	x̄	2.
3	3	0.2	Σx	2.
4	4	0.1	Σx²	5.
5			sx := sn-...	#UNDER...
6			σx := σn...	1.

D6 =1.

2 a



Mode $(X) = 0$, the part of parabola is opened downwards and it has its vertex at $\left(0, \frac{3}{4}\right)$.

Median, $\int_0^m \left(\frac{3}{4} - \frac{3}{16}x^2\right)dx = \frac{1}{2} \Rightarrow$

$$\frac{3}{4}m - \frac{1}{16}m^3 = \frac{1}{2} \Rightarrow m = 0.695$$

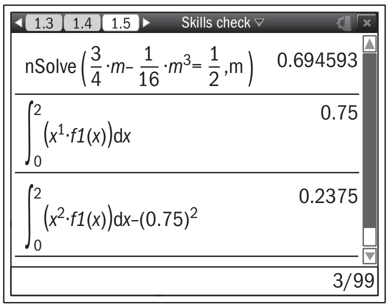
$$\mu = E(X) = \int_0^2 \left(\frac{3}{4}x - \frac{3}{16}x^3\right)dx$$

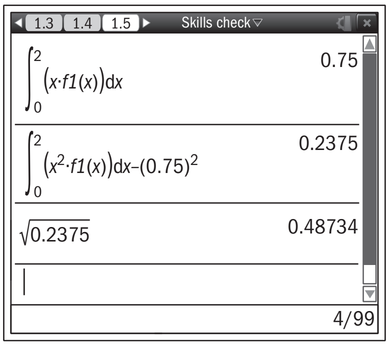
$$= \left[\frac{3}{8}x^2 - \frac{3}{64}x^4\right]_0^2 = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\int_0^2 \left(\frac{3}{4}x^2 - \frac{3}{16}x^4\right)dx - \left(\frac{3}{4}\right)^2}$$

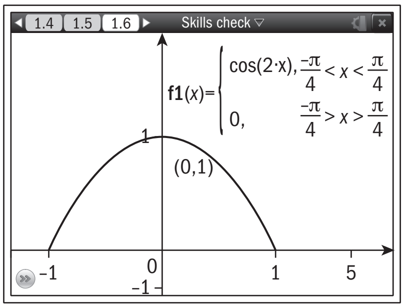
$$= \sqrt{\left[\frac{x^3}{4} - \frac{3x^5}{80}\right]_0^2 - \frac{9}{16}} = \sqrt{2 - \frac{6}{5} - \frac{9}{16}}$$

$$= \sqrt{\frac{19}{80}} = 0.487$$





b

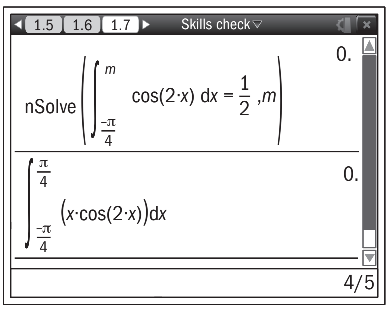


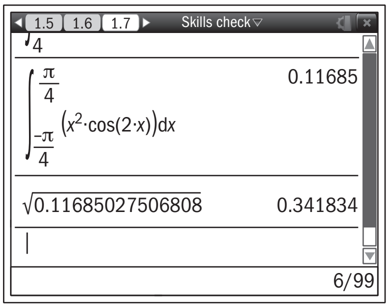
Mode $(X) = 0$, it has the maximum point at (0, 1).

Median, $\int_{-\frac{\pi}{4}}^{m} f(x)dx = \frac{1}{2} \Rightarrow m = 0$

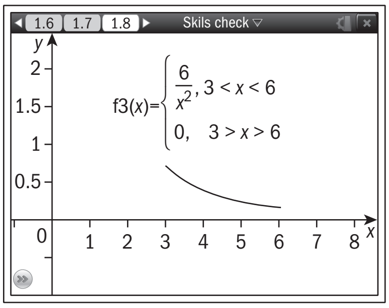
$$\mu = \mathrm{E}(X) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} xf(x)dx = 0$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^2 f(x) - \mu^2} = 0.342$$





c

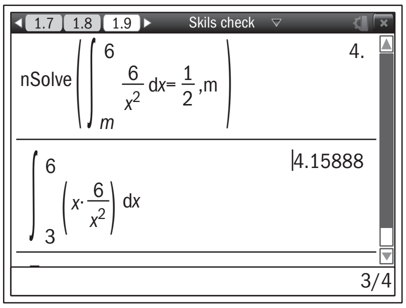


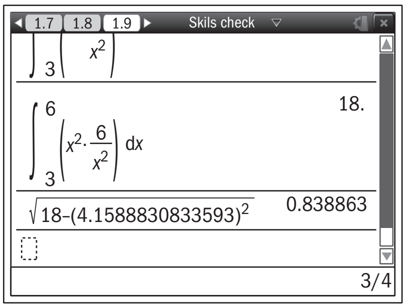
Mode $(X) = 3$ because it is a decreasing function on the interval [3, 6].

Median, $\int_0^m f(x)dx = \frac{1}{2} \Rightarrow m = 4$

$$\mu = \mathrm{E}(X) = \int_3^6 xf(x)dx = 4.16$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\int_3^6 x^2 f(x) - \mu^2} = 0.839$$





3 a $1 - 0.5 + 0.25 - 0.125 + \ldots \Rightarrow u_1 = 1,\ r = -0.5$

$$\text{Sum} = \frac{1}{1-(-0.5)} = \frac{1}{1.5} = \frac{2}{3}$$

b $\sqrt{2} + 1 + \frac{\sqrt{2}}{2} + \frac{1}{2} \ldots \Rightarrow u_1 = \sqrt{2},\ r = \frac{1}{\sqrt{2}}$

$$\text{Sum} = \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} = \frac{2}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 2 + 2\sqrt{2}$$

4 **a** $f(x) = \frac{1}{2-x},\ x \neq 2$

$\Rightarrow f'(x) = \frac{-1 \times -1}{(2-x)^2} = \frac{1}{(2-x)^2},\ x \neq 2$

$\Rightarrow \int f(x)dx = \int \frac{dx}{2-x} = -\ln(2-x) + c,\ x < 2$

b $f(x) = e^{3x+1} \Rightarrow f'(x) = 3e^{3x+1}$

$\Rightarrow \int f(x)dx = \int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + c$

c $f(x) = \frac{3}{2}\sin\frac{\pi-2x}{3}$

$\Rightarrow f'(x) = \frac{3}{2}\cos\frac{\pi-2x}{3} \times \left(-\frac{2}{3}\right) = -\cos\frac{\pi-2x}{3}$

$\Rightarrow \int f(x)dx = \int \frac{3}{2}\sin\frac{\pi-2x}{3}dx$

$= \frac{3}{2} \times -\frac{1}{-\frac{2}{3}}\cos\frac{\pi-2x}{3} + c = \frac{9}{4}\cos\frac{\pi-2x}{3} + c$

d $f(x) = (x^2-2)^2$

$\Rightarrow f'(x) = 2(x^2-2) \times 2x = 4x(x^2-2)$

$\Rightarrow \int f(x)dx = \int (x^2-2)^2 dx$

$= \int (x^4 - 4x^2 + 4)dx = \frac{x^5}{5} - \frac{4x^3}{3} + 4x + c$

Exercise 1A

1 **a** $\frac{k}{12} + \frac{1+k}{12} + \frac{2+k}{12} = 1 \Rightarrow 3 + 3k = 12 \Rightarrow k = 3$

b

$X = x$	0	1	2
$F(x)$	$\frac{3}{12}$	$\frac{7}{12}$	$\frac{12}{12}$

$$\Rightarrow F(x) = \begin{cases} 0,\ x < 0 \\ \frac{x^2+7x+6}{24},\ x = 0, 1, 2 \\ 1,\ x > 2 \end{cases}$$

2 **a** $\frac{a-2}{20} + \frac{a-4}{20} + \frac{a-6}{20} + \frac{a-8}{20} = 1$

$\Rightarrow 4a - 20 = 20 \Rightarrow a = 10$

b

$X = x$	1	2	3	4
$F(x)$	$\frac{8}{20}$	$\frac{14}{20}$	$\frac{18}{20}$	$\frac{20}{20}$

$$\Rightarrow F(x) = \begin{cases} 0,\ x < 1 \\ \frac{9x-x^2}{20},\ x = 1, 2, 3, 4 \\ 1,\ x > 4 \end{cases}$$

c $P(X \le 2) = F(2) = \frac{14}{20} = \frac{7}{10}$

3 **a** $P(X = 0) = F(0) = \frac{1}{6}$,

$P(X = 1) = F(1) - F(0) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6}$,

$P(X = 2) = F(2) - F(1) = 1 - \frac{1}{2} = \frac{1}{2}$

$= \frac{3}{6} \Rightarrow P(X = x) = \begin{cases} \frac{x+1}{6},\ x = 0, 1, 2 \\ 0, \text{ otherwise} \end{cases}$

b The modal value of the variable X is 2 since it has the highest probability, $P(X = 2) = \frac{1}{2}$.

4 **a** $P(X = 1) = F(1) = \frac{1}{25}$

$P(X = 3) = F(3) - F(1) = \frac{3}{25}$

$P(X = 5) = F(5) - F(3) = \frac{5}{25}$

$P(X = 7) = F(7) - F(5) = \frac{7}{25}$

$P(X = 9) = F(9) - F(7) = \frac{9}{25}$

$$P(X = x) = \begin{cases} \frac{x}{25},\ x = 1, 3, 5, 7, 9 \\ 0, \text{ otherwise} \end{cases}$$

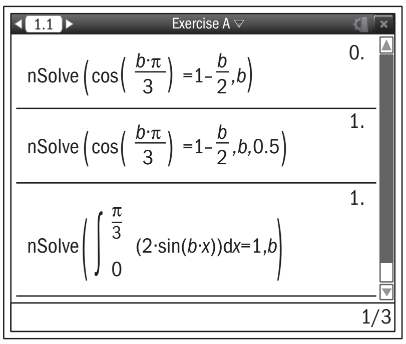
b Median, $m = 7$ since

$F(5) = \frac{9}{25} < \frac{1}{2}$ and $F(7) = \frac{16}{25} > \frac{1}{2}$

5 **a** $\int_0^{\frac{\pi}{3}} 2\sin bx\,dx = 1 \Rightarrow 2\left[-\frac{1}{b}\cos bx\right]_0^{\frac{\pi}{3}} = 1$

$\Rightarrow \cos\left(\frac{b\pi}{3}\right) - \cos 0 = -\frac{b}{2}$

$\Rightarrow \cos\left(\frac{b\pi}{3}\right) = 1 - \frac{b}{2} \Rightarrow b = 1$



b $F(x) = \int_{-\infty}^{x} f(t)dt \Rightarrow F(x) = \int_0^x 2\sin t\,dt$

$= [-2\cos t]_0^x = -2\cos x + 2$

$$\Rightarrow F(x) = \begin{cases} 0,\ x < 0 \\ 2 - 2\cos x,\ 0 \le x \le \frac{\pi}{3} \\ 1,\ x > \frac{\pi}{3} \end{cases}$$

c $P\left(X \geq \frac{\pi}{6}\right) = 1 - P\left(X \leq \frac{\pi}{6}\right) = 1 - F\left(\frac{\pi}{6}\right)$

$= 1 - 2 + 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3} - 1 = 0.732$

6 **a** Since the function is not defined for the values of -2 and 2 we need to use limits.

$$\lim_{b \to 2}\left(\int_{-b}^{b} \frac{dx}{\pi\sqrt{4 - x^2}}\right) = \lim_{b \to 2}\left[\frac{1}{\pi}\arcsin\frac{x}{2}\right]_{-b}^{b}$$

$$= \frac{1}{\pi}\left[\arcsin\frac{x}{2}\right]_{-2}^{2} = \frac{1}{\pi}(\arcsin 1 - \arcsin(-1))$$

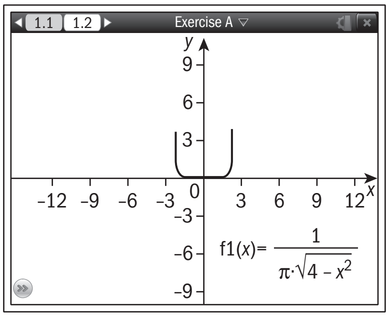
$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1$$

The function is well defined if $\int_{-\infty}^{+\infty} f(x)dx = 1$

b The modal value is the value of the random variable X where probability reaches its maximum.

$$f'(x) = \frac{1}{\pi}\left(-\frac{1}{2}\right)(4 - x^2)^{-\frac{3}{2}}(-2x)$$

$$= \frac{x}{\pi(4 - x^2)^{\frac{3}{2}}}, \quad f'(x) = 0 \Rightarrow x = 0$$



By looking at the graph of the probability function we notice that the graph has only one extreme point and that is the minimum point at 0, therefore the modal value doesn't exist.

Exercise 1B

1 **a** $X \sim \text{Geo}(0.6) \Rightarrow P(X = 2) = 0.4 \times 0.6 = 0.24$

b $X \sim \text{Geo}(0.14) \Rightarrow P(X = 3)$

$= 0.86^2 \times 0.14 = 0.104$

c $X \sim \text{Geo}(0.5) \Rightarrow P(X = 4)$

$= 0.5^3 \times 0.5 = 0.0625$

d $X \sim \text{Geo}(0.88) \Rightarrow P(X = 5)$

$= 0.12^4 \times 0.88 = 0.000182$

1.1 CH1 Exercise B

geomPdf(0.6,2)	0.24
geomPdf(0.14,3)	0.103544
geomPdf(0.5,4)	0.0625
geomPdf(0.88,5)	0.000182

4/99

2 **a** $X \sim \text{Geo}(0.25) \Rightarrow P(X \leq 4) = 0.684$

b $X \sim \text{Geo}(0.7) \Rightarrow P(X > 6)$

$= 1 - P(X \leq 6) = 0.000729$

c $X \sim \text{Geo}(0.3) \Rightarrow P(5 \leq X \leq 7)$

$= P(X \leq 7) - P(X \leq 4) = 0.158$

d $X \sim \text{Geo}(0.991) \Rightarrow P(1 < X \leq 7)$

$= P(X \leq 7) - P(X \leq 1) = 0.009$

1.1 1.2 CH1 Exercise B

geomCdf(0.25,1,4)	0.683594
1-geomCdf(0.7,1,6)	0.000729
geomCdf(0.3,1,7)-geomCdf(0.3,1,4)	0.157746
geomCdf(0.991,1,7)-geomCdf(0.991,1,1)	0.009

4/99

3 $X \sim \text{Geo}(0.73) \Rightarrow P(X \leq 3) = 0.980$

4 $X \sim \text{Geo}(0.92)$

a $P(X = 5) = 0.0000377$

b $P(X \geq 4) = 1 - P(X \leq 3) = 0.000512$

5 $X \sim \text{Geo}(0.15)$

1.1 1.2 1.3 CH1 Exercise B

geomCdf(0.73,1,3)	0.980317
geomPdf(0.92,5)	0.000038
1-geomCdf(0.92,1,3)	0.000512
geomPdf(0.15,4)	0.092119
1-geomCdf(0.15,1,6)	0.37715

5/99

a $P(X = 4) = 0.0921$

b $P(X \geq 7) = 1 - P(X \leq 6) = 0.377$

6 $X \sim \text{Geo}(p) \Rightarrow P(X > k) = 1 - P(X \leq k)$

$$= 1 - \sum_{n=k+1}^{\infty} q^{n-1}p = 1 - p\underbrace{\sum_{n=1}^{k} q^{n-1}}_{\text{inf. geom. seq.}}$$

$$= 1 - p\frac{1 - q^k}{1 - q} = 1 - p\frac{1 - q^k}{p} = 1 - (1 - q^k) = q^k$$

OR

$X \sim \text{Geo}(p) \Rightarrow P(X > k)$

$$= \sum_{n=k+1}^{\infty} q^{n-1}p = q^k p \times \underbrace{\sum_{n=1}^{k} q^{n-1}}_{\text{inf. geom. seq.}} = q^k p\frac{1}{1 - q} = q^k p\frac{1}{p} = q^k$$

Investigation 1

$X \sim \text{Geo}(p)$

a $p = 0.4 \Rightarrow \text{P}(X > 5 \mid X > 3)$

$= \dfrac{\text{P}((X > 5) \cap (X > 3))}{\text{P}(X > 3)} = \dfrac{1 - \text{P}(X \le 5)}{1 - \text{P}(X \le 3)} = 0.36$

$\text{P}(X > 2) = 1 - \text{P}(X \le 2) = 0.36$

b $p = 0.7 \Rightarrow \text{P}(X > 6 \mid X > 2)$

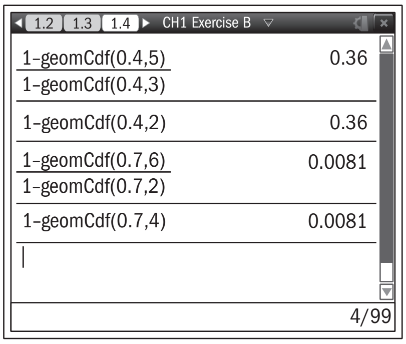
$= \dfrac{\text{P}((X > 6) \cap (X > 2))}{\text{P}(X > 2)} = \dfrac{1 - \text{P}(X \le 6)}{1 - \text{P}(X \le 2)} = 0.0081$

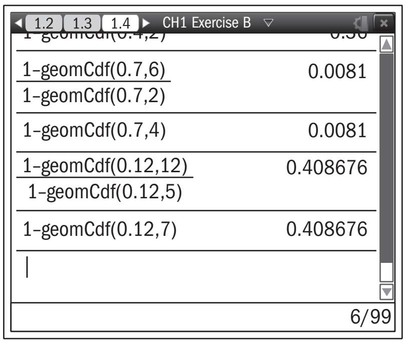
$\text{P}(X > 4) = 1 - \text{P}(X \le 4) = 0.0081$

c $p = 0.12 \Rightarrow \text{P}(X > 12 \mid X > 5)$

$= \dfrac{\text{P}((X > 12) \cap (X > 5))}{\text{P}(X > 5)} = \dfrac{1 - \text{P}(X \le 12)}{1 - \text{P}(X \le 5)} = 0.409$

$\text{P}(X > 7) = 1 - \text{P}(X \le 7) = 0.409$





$X \sim \text{Geo}(p);\ m, n \in \mathbb{Z}^+;\ m < n$

$\Rightarrow \text{P}(X > n \mid X > m) = \dfrac{\text{P}((X > n) \cap (X > m))}{\text{P}(X > m)}$

$= \dfrac{\text{P}(X > n)}{\text{P}(X > m)} = \dfrac{q^n}{q^m} = q^{n-m} = \text{P}(X > n - m)$

Exercise 1C

1 For 1B, Question 1:

a $X \sim \text{Geo}(0.6) \Rightarrow \text{E}(X) = \dfrac{1}{0.6} = 1.67,$

$\text{Var}(X) = \dfrac{0.4}{0.6^2} = 1.11$

b $X \sim \text{Geo}(0.14) \Rightarrow \text{E}(X) = \dfrac{1}{0.14} = 7.14,$

$\text{Var}(X) = \dfrac{0.86}{0.14^2} = 43.9$

c $X \sim \text{Geo}(0.5) \Rightarrow \text{E}(X) = \dfrac{1}{0.5} = 2,$

$\text{Var}(X) = \dfrac{0.5}{0.5^2} = 2$

d $X \sim \text{Geo}(0.88) \Rightarrow \text{E}(X) = \dfrac{1}{0.88} = 1.14,$

$\text{Var}(X) = \dfrac{0.12}{0.88^2} = 0.155$

For 1B, Question 2:

a $X \sim \text{Geo}(0.25) \Rightarrow \text{E}(X) = \dfrac{1}{0.25} = 4,$

$\text{Var}(X) = \dfrac{0.75}{0.25^2} = 12$

b $X \sim \text{Geo}(0.7) \Rightarrow \text{E}(X) = \dfrac{1}{0.7} = 1.43,$

$\text{Var}(X) = \dfrac{0.3}{0.7^2} = 0.612$

c $X \sim \text{Geo}(0.3) \Rightarrow \text{E}(X) = \dfrac{1}{0.3} = 3.33,$

$\text{Var}(X) = \dfrac{0.7}{0.3^2} = 7.78$

d $X \sim \text{Geo}(0.991) \Rightarrow \text{E}(X) = \dfrac{1}{0.991} = 1.01,$

$\text{Var}(X) = \dfrac{0.009}{0.991^2} = 0.00916$

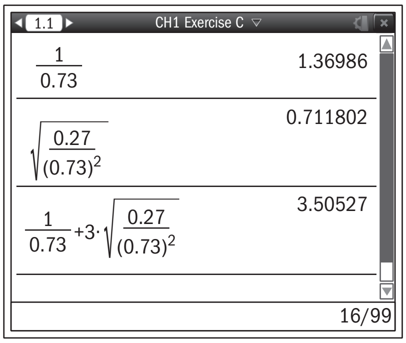
2 $X \sim \text{Geo}(0.73)$

a $\text{E}(X) = \dfrac{1}{0.73} = 1.37$

b $\sigma = \sqrt{\text{Var}(X)} = \sqrt{\dfrac{0.27}{0.73^2}} = 0.712$

$x_{\max} = \text{E}(X) + 3\sigma = \dfrac{1}{0.73} + 3 \times \sqrt{\dfrac{0.27}{0.73^2}} = 3.51$

Mario must make four shots to destroy the balloon.

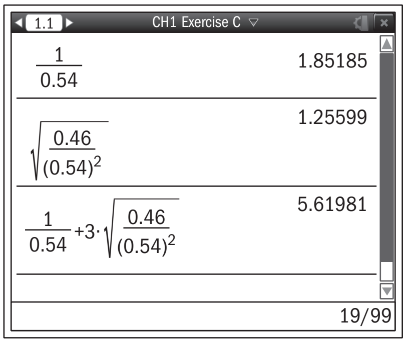


3 $X \sim \text{Geo}(0.54)\quad \text{E}(X) = \dfrac{1}{0.54} = 1.85$

$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\dfrac{0.46}{0.54^2}} = 1.26$

$x_{\max} = \text{E}(X) + 3\sigma = \dfrac{1}{0.54} + 3 \times \sqrt{\dfrac{0.46}{0.54^2}} = 5.62$

Therefore, we must select 6 students at random to ensure that one of them will be familiar with the election procedure.



Exercise 1D

1 a $X \sim \mathrm{NB}(1, 0.2) \Rightarrow \mathrm{P}(X = 2)$

$$= \binom{2-1}{1-1} 0.8 \times 0.2 = 0.16$$

b $X \sim \mathrm{NB}(3, 0.5) \Rightarrow \mathrm{P}(X = 4)$

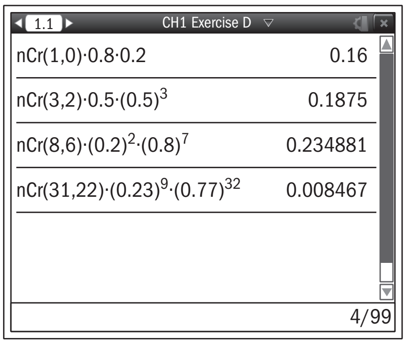
$$= \binom{4-1}{3-1} 0.5 \times 0.5^3 = 0.188$$

c $X \sim \mathrm{NB}(7, 0.8) \Rightarrow \mathrm{P}(X = 9)$

$$= \binom{9-1}{7-1} 0.2^2 \times 0.8^7 = 0.235$$

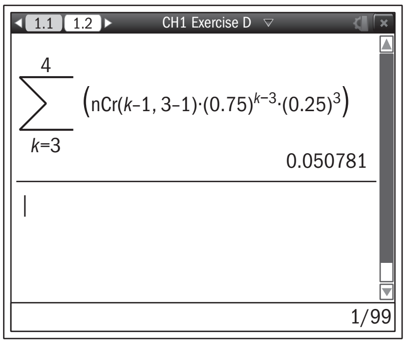
d $X \sim \mathrm{NB}(23, 0.77) \Rightarrow \mathrm{P}(X = 32)$

$$= \binom{32-1}{23-1} 0.23^9 \times 0.77^{32} = 0.00847$$



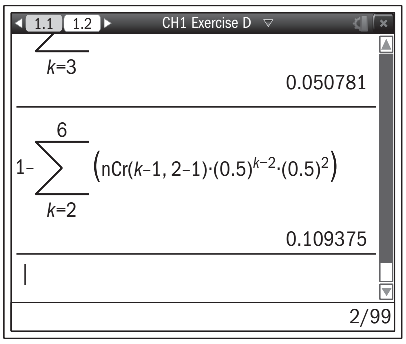
2 a $X \sim \mathrm{NB}(0.25, 3) \Rightarrow \mathrm{P}(X \le 4)$

$$= \mathrm{P}(X = 3) + \mathrm{P}(X = 4) = \frac{13}{256} = 0.508$$



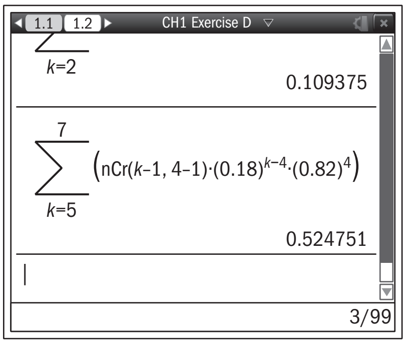
b $X \sim \mathrm{NB}(0.5, 2) \Rightarrow \mathrm{P}(X > 6)$

$$= 1 - \sum_{k=2}^{6} \mathrm{P}(X = k) = \frac{7}{64} = 0.109$$



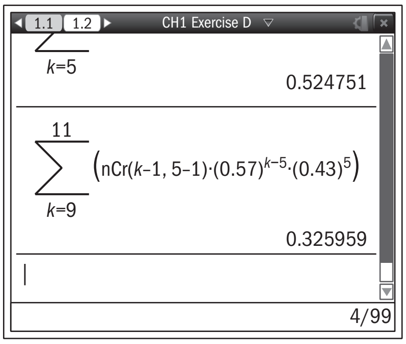
c $X \sim \mathrm{NB}(0.82, 4) \Rightarrow \mathrm{P}(5 \le X \le 7)$

$$= \sum_{k=5}^{7} \mathrm{P}(X = k) = 0.525$$



d $X \sim \mathrm{NB}(0.43, 5) \Rightarrow \mathrm{P}(8 < X \le 11)$

$$= \sum_{k=9}^{11} \mathrm{P}(X = k) = 0.326$$



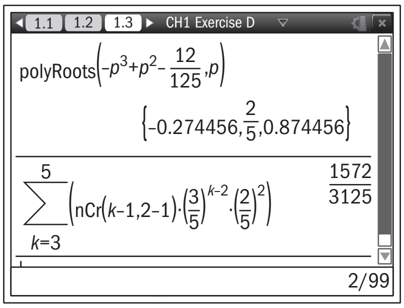
3 a $P(X = 3) = \frac{24}{125} \Rightarrow \binom{3-1}{2-1}(1-p) \times p^2$

$$= \frac{24}{125} \Rightarrow p^2 - p^3 = \frac{12}{125} \Rightarrow \begin{cases} p_1 = -0.274 \\ p_2 = \frac{2}{5} \\ p_3 = 0.874 \end{cases}$$

Thus $p = \frac{2}{5} = 0.4$

b $X \sim \text{NB}\left(2, \frac{2}{5}\right) \Rightarrow \text{P}(3 \le X \le 5)$

$$= \sum_{k=3}^{5} \text{P}(X = k) = \frac{1572}{3125} = 0.503$$



4 a $X \sim \text{NB}\left(2, \frac{1}{4}\right)$

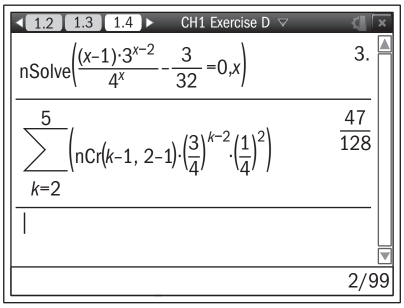
b $X \sim \text{NB}\left(2, \frac{1}{4}\right) \Rightarrow \text{P}(X = x) = \frac{3}{32}$

$$\Rightarrow \binom{x-1}{2-1}\left(\frac{3}{4}\right)^{x-2} \times \left(\frac{1}{4}\right)^2 = \frac{3}{32}$$

$$\Rightarrow (x-1)\frac{3^{x-2}}{4^x} = \frac{3}{32} \Rightarrow x = 3$$

c $X \sim \text{NB}\left(2, \frac{1}{4}\right) \Rightarrow \text{P}(X \le 5)$

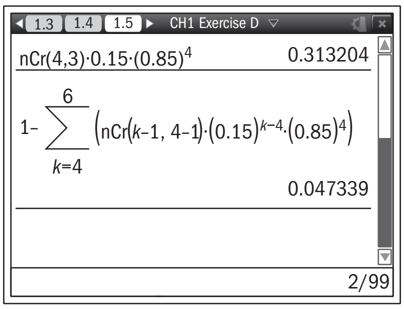
$$= \sum_{k=2}^{5} \text{P}(X = k) = \frac{47}{128}$$



5 a $X \sim \text{NB}(0.85, 4) \Rightarrow \text{P}(X = 5) = 0.313$

b $X \sim \text{NB}(0.85, 4) \Rightarrow \text{P}(X \ge 7)$

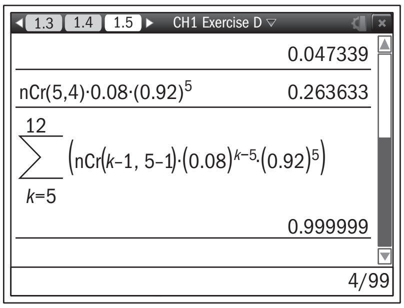
$$= 1 - \text{P}(X \le 6) = 0.0473$$



6 a $X \sim \text{NB}(0.92, 5) \Rightarrow \text{P}(X = 6) = 0.264$

b $X \sim \text{NB}(0.92, 5) \Rightarrow \text{P}(X \le 12) = 0.999999 \approx 1$

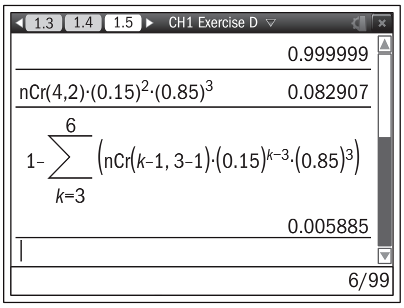
We can say that it is almost certain that he will not need to interview more than a dozen students.



7 a $X \sim \text{NB}(0.85, 3) \Rightarrow \text{P}(X = 5) = 0.0829$

b $X \sim \text{NB}(0.85, 3) \Rightarrow \text{P}(X \ge 7)$

$$= 1 - \text{P}(X \le 6) = 0.589$$



Exercise 1E

1 $\text{P}(X = x) = \binom{4}{x}\underbrace{\left(\frac{1}{2}\right)^x}_{heads}\underbrace{\left(\frac{1}{2}\right)^{4-x}}_{tails} = \binom{4}{x}\left(\frac{1}{2}\right)^4$

x_i	0	1	2	3	4
p_i	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$G(t) = \frac{1}{16} + \frac{1}{4}t + \frac{3}{8}t^2 + \frac{1}{4}t^3 + \frac{1}{16}t^4$$

2

x_i	1	2	3	…	k	…
p_i	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{25}{216}$	…	$\underbrace{\left(\frac{5}{6}\right)^{k-1}}_{\text{not one}} \times \underbrace{\frac{1}{6}}_{\text{one}} = \frac{5^{k-1}}{6^k}$	…

$$G(t) = \frac{1}{6}t + \frac{5}{36}t^2 + \frac{25}{216}t^3 + \dots$$

$$+ \frac{5^{k-1}}{6^k}t^k + \dots = \frac{1}{6}t\underbrace{\sum_{k=0}^{\infty}\left(\frac{5}{6}t\right)^k}_{\text{inf. geom. sequence}}$$

$$= \frac{1}{6}t \times \frac{1}{1-\frac{5}{6}t} = \frac{1}{\cancel{6}}t \times \frac{\cancel{6}}{6-5t} = \frac{t}{6-5t}$$

$$\left|\frac{5}{6}t\right| < 1 \Rightarrow -1 < \frac{5}{6}t < 1 \Rightarrow -\frac{6}{5} < t < \frac{6}{5}$$

3 a $G(t) = \frac{2}{3-t} = \frac{\frac{2}{3}}{1-\frac{1}{3}t} = \frac{2}{3} + \frac{2}{9}t + \frac{2}{27}t^2$

$$+ \frac{2}{81}t^3 + \dots + \frac{2}{3^{k+1}}t^k + \dots \Rightarrow \text{P}(X = 0) = \frac{2}{3}$$

b $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$

c $P(X \ge 3) = 1 - P(X \le 2)$

$$= 1 - \left(\frac{2}{3} + \frac{2}{9} + \frac{2}{27}\right) = 1 - \frac{26}{27} = \frac{1}{27}$$

d $P(X \ge k) = 1 - P(X \le k - 1)$

$$= 1 - \left(\frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^k}\right)$$

$$= 1 - \frac{\frac{2}{3}\left(1 - \frac{1}{3^k}\right)}{1 - \frac{1}{3}} = 1 - 1 + \frac{1}{3^k} = \frac{1}{3^k}$$

4 a $X \sim B(1, p)$

x_i	0	1
p_i	$1 - p$	p

$G(t) = 1 - p + pt$

$G'(t) = p, G''(t) = 0$

$E(X) = G'(1) = p,$

$\text{Var}(X) = G''(1) + G'(1)(1 - G'(1))$

$$= 0 + p(1 - p) = p(1 - p)\,[= pq]$$

b $X \sim NB(r, p), G(t) = \left(\frac{pt}{1 - qt}\right)^r$

$$G'(t) = r\left(\frac{pt}{1 - qt}\right)^{r-1} \frac{p(1 - qt) - pt \times (-q)}{(1 - qt)^2}$$

$$= rp\left(\frac{pt}{1 - qt}\right)^{r-1} \frac{1}{(1 - qt)^2} = rp^r t^{r-1}(1 - qt)^{-(r+1)}$$

$$G'(1) = rp^r p^{-(r+1)} = \frac{r}{p}$$

$$G''(t) = rp^r(r - 1)t^{r-2}(1 - qt)^{-(r+1)}$$

$$+ rp^r t^{r-1} \times (-(r + 1))(1 - qt)^{-(r+2)} \times (-q)$$

$$= rp^r t^{r-2}(1 - qt)^{-(r+2)}((r - 1)(1 - qt) + (r + 1)qt)$$

$$G''(1) = rp^r \times p^{-r-2}((r - 1)p + (r + 1)q)$$

$$= \frac{r}{p^2}\left(-p + \overbrace{rp + rq}^{r} + q\right) = \frac{r}{p^2}(-p + r + q)$$

$E(X) = G'(1) = \frac{r}{p},$

$\text{Var}(X) = G''(1) + G'(1)(1 - G'(1))$

$$= \frac{r}{p^2}(-p + r + q) + \frac{r}{p}\left(1 - \frac{r}{p}\right)$$

$$= \frac{-\cancel{rp} + \cancel{r^2} + rq + \cancel{rp} - \cancel{r^2}}{p^2} = \frac{rq}{p^2}$$

5 For question 1:

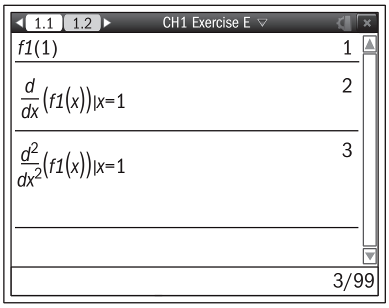
$$G(t) = \frac{1}{16} + \frac{1}{4}t + \frac{3}{8}t^2 + \frac{1}{4}t^3 + \frac{1}{16}t^4,$$

$$G'(t) = \frac{1}{4} + \frac{3}{4}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 \Rightarrow G'(1) = 2$$

$$G''(t) = \frac{3}{4} + \frac{3}{2}t + \frac{3}{4}t^2 \Rightarrow G''(1) = 3$$

$E(X) = G'(1) = 2; \text{Var}(X) = G''(1)$

$$+ G'(1)(1 - G'(1)) = 3 + 2(1 - 2) = 3 - 2 = 1$$



For question 2:

$$G(t) = \frac{t}{6 - 5t}, G'(t) = \frac{6 - 5t - t \times (-5)}{(6 - 5t)^2}$$

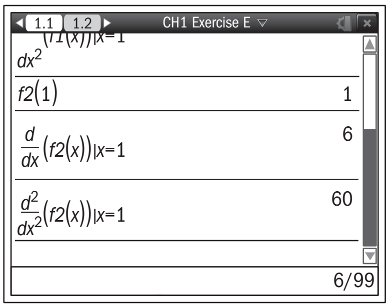
$$= 6(6 - 5t)^{-2} \Rightarrow G'(1) = 6$$

$$G''(t) = -12(6 - 5t)^{-3} \times (-5)$$

$$= 60(6 - 5t)^{-2} \Rightarrow G''(1) = 60$$

$E(X) = G'(1) = 6; \text{Var}(X) = G''(1)$

$$+ G'(1)(1 - G'(1)) = 60 + 6(1 - 6) = 60 - 30 = 30$$



For question 3:

$$G(t) = \frac{2}{3 - t}, G'(t) = 2(3 - t)^{-2}$$

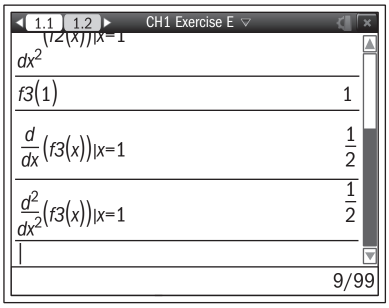
$$\Rightarrow G'(1) = 2(3 - 1)^{-2} = \frac{1}{2}$$

$$G''(t) = 4(3 - t)^{-3} \Rightarrow G''(1) = 4(3 - 1)^{-3} = \frac{1}{2}$$

$E(X) = G'(1) = \frac{1}{2},$

$\text{Var}(X) = G''(1) + G'(1)(1 - G'(1))$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$



6 a $P(X=1)=\frac{4}{7}$; $P(X=4)=\frac{3}{7}\times\frac{1}{3}\times\frac{3}{7}\times\frac{2}{3}=\frac{2}{49}$

b $G(t)=\frac{4}{7}t+\frac{3}{7}\times\frac{2}{3}t^2+\frac{3}{7}\times\frac{1}{3}\times\frac{4}{7}t^3$

$+\frac{3}{7}\times\frac{1}{3}\times\frac{3}{7}\times\frac{2}{3}t^4+\frac{3}{7}\times\frac{1}{3}\times\frac{3}{7}\times\frac{1}{3}\times\frac{4}{7}t^5$

$+\frac{3}{7}\times\frac{1}{3}\times\frac{3}{7}\times\frac{1}{3}\times\frac{3}{7}\times\frac{2}{3}t^6+\ldots$

$=\frac{4}{7}t\left(1+\frac{1}{7}t^2+\frac{1}{49}t^4+\ldots\right)$

$+\frac{2}{7}t^2\left(1+\frac{1}{7}t^2+\frac{1}{49}t^4+\ldots\right)$

$=\left(\frac{4}{7}t+\frac{2}{7}t^2\right)\left(1+\frac{1}{7}t^2+\frac{1}{49}t^4+\ldots\right)$

$=\frac{4t+2t^2}{7}\times\frac{1}{1-\frac{1}{7}t^2}=\frac{4t+2t^2}{\cancel{7}}\times\frac{\cancel{7}}{7-t^2}=\frac{4t+2t^2}{7-t^2}$

Verifying: $G(1)=\frac{4+2}{7-1}=\frac{6}{6}=1$

c $G(t)=\frac{4t+2t^2}{7-t^2}$, $G'(t)=\frac{(4+4t)(7-t^2)+2t(4t+2t^2)}{(7-t^2)^2}$

$=\frac{4(7+7-t^2-\cancel{t^3}+2t^2+\cancel{t^3})}{(7-t^2)^2}=\frac{4(7+7t+t^2)}{(7-t^2)^2}$

$E(X)=G'(1)=\frac{4(7+7+1)}{(7-1)^2}=\frac{5}{3}$

The expected number of shots before the game is over is 2.

d $G'(t)=4(7+7t+t^2)(7-t^2)^{-2}$,

$G''(t)=4(7+2t)(7-t^2)^{-2}$

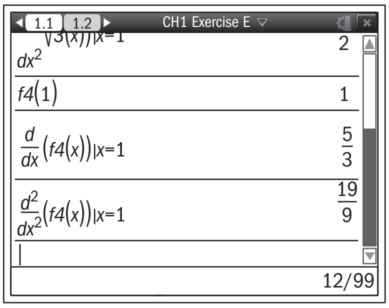
$-2(7-t^2)^{-3}\times(-2t)\times4(7+7t+t^2)$

$=\frac{4((7+2t)(7-t^2)+4t(7+7t+t^2))}{(7-t^2)^3}$

$=\frac{4(49+14t-7t^2-2t^3+28t+28t^2+4t^3)}{(7-t^2)^3}$

$=\frac{4(49+42t+21t^2+2t^3)}{(7-t^2)^3}$

$\Rightarrow G''(1)=\frac{4(49+42+21+2)}{(7-1)^3}=\frac{19}{9}$



$Var(X)=G''(1)+G'(1)(1-G'(1))$

$=\frac{19}{9}+\frac{5}{3}\left(1-\frac{5}{3}\right)=\frac{19}{9}-\frac{10}{9}=1$

$x_{max}=E(X)+3\sigma=\frac{5}{3}+3\times\sqrt{1}=\frac{14}{3}=4.67.$

Therefore, the maximum number of shots to be made before the game is over is 5.

Exercise 1F

1 a $G_{X+Y}(t)=G_X(t)\times G_Y(t)=\left(\frac{1+3t}{4}\right)^2$

$\times\left(\frac{2+t}{3}\right)^2=\left(\frac{2+7t+3t^2}{12}\right)^2$

b $G_{X+Y}(t)=\left(\frac{2+7t+3t^2}{12}\right)^2=\frac{4}{144}+\frac{28}{144}t+\ldots$

$\Rightarrow P(X+Y\le1)=\frac{4}{144}+\frac{28}{144}=\frac{32}{144}=\frac{2}{9}$

c $G'_{X+Y}(t)=\frac{1}{144}\times2(2+7t+3t^2)(7+6t)$

$\Rightarrow E(X+Y)=G'_{X+Y}(1)=\frac{1}{72}\times12\times13=\frac{13}{6}$

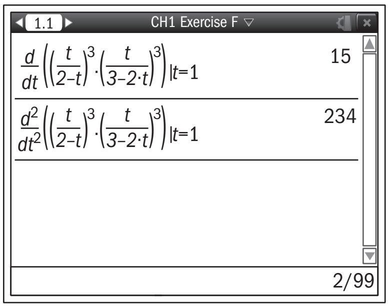
2 a $G_{X+Y}(t)=G_X(t)\times G_Y(t)=\left(\frac{t}{2-t}\right)^3\times\left(\frac{t}{3-2t}\right)^3$

$=\left(\frac{t^2}{6-7t+2t^2}\right)^3$

b $E(X+Y)=G'(1)=15$

c $Var(X+Y)=G''(1)+G'(1)(1-G'(1))$

$=234+15\times(-14)=234-210=24$



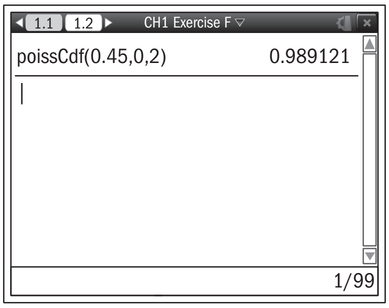
3 a $G_X(t)=e^{0.25(t-1)}$, $G_Y(t)=e^{0.15(t-1)}$, $G_Z(t)=e^{0.05(t-1)}$

b $G_{X+Y+Z}(t)=G_X(t)\times G_Y(t)\times G_Z(t)$

$=e^{0.25(t-1)}\times e^{0.15(t-1)}\times e^{0.05(t-1)}=e^{0.45(t-1)}$

$G_{X+Y+Z}(t)=e^{0.45(t-1)}\Rightarrow X+Y+Z\sim Po(0.45)$

$P(X+Y+Z\le2)=0.989$



4 $X \sim \text{Geo}(p),\ G_X(t) = \dfrac{pt}{1-qt}$

$Y \sim \text{NB}(r, p),\ Y = \underbrace{X + X + \ldots + X}_{r \text{ terms(sucesses)}} \Rightarrow G_Y(t)$

$= G_{\underbrace{X+X+\ldots+X}_{r \text{ terms}}}(t) = \underbrace{G_X(t) \times G_X(t) \times \ldots \times G_X(t)}_{r \text{ factors}}$

$= \underbrace{\dfrac{pt}{1-qt} \times \dfrac{pt}{1-qt} \times \ldots \times \dfrac{pt}{1-qt}}_{r \text{ factors}} = \left(\dfrac{pt}{1-qt}\right)^r$

5 a $\left.\begin{array}{l} X_1 \sim \text{B}(n_1, p),\ G_{X_1}(t) = (q + pt)^{n_1} \\ X_2 \sim \text{B}(n_2, p),\ G_{X_2}(t) = (q + pt)^{n_2} \end{array}\right\} \Rightarrow G_{X_1+X_2}(t)$

$= G_{X_1}(t) \times G_{X_2}(t) = (q + pt)^{n_1} \times (q + pt)^{n_2}$

$= (q + pt)^{n_1+n_2} \Rightarrow X_1 + X_2 \sim \text{B}(n_1 + n_2, p)$

b 1st Base

For $n = 1$, the statement is trivial: $X_1 \sim \text{B}(n_1, p)$

For $n = 2$, in part **a** we proved that

$X_1 + X_2 \sim \text{B}(n_1 + n_2, p)$

2nd Assumption

Let's assume that for $n = k$ this statement is true:

$\sum_{i=1}^{k} X_i \sim \text{B}\left(\sum_{i=1}^{k} n_i, p\right)$

3rd Step

For $n = k + 1$, $\sum_{i=1}^{k+1} X_i = \sum_{i=1}^{k} X_i + X_{k+1}$,

by the assumption the first term has a binomial distribution and by the given proposition in the question $X_{k+1} \sim \text{B}(n_{k+1}, p)$. Now we use the fact from part **a** that the sum of two binomial variables with the same probability p satisfies the following:

$\left.\begin{array}{l} \sum_{i=1}^{k} X_i \sim \text{B}\left(\sum_{i=1}^{k} n_i, p\right) \\ X_{k+1} \sim \text{B}(n_{k+1}, p) \end{array}\right\} \Rightarrow \sum_{i=1}^{k} X_i + X_{k+1} \sim$

$\text{B}\left(\sum_{i=1}^{k} n_i + n_{k+1}, p\right) \Rightarrow \sum_{i=1}^{k+1} X_i \sim \text{B}\left(\sum_{i=1}^{k+1} n_i, p\right)$

4th Conclusion

Now we can conclude that $Y = \sum_{i=1}^{k} X_i \sim \text{B}\left(\sum_{i=1}^{k} n_i, p\right)$

for all the positive integer values of k.

Review exercise

1 a $G(t) = \dfrac{4t}{5-t} = \dfrac{4}{5}t \times \dfrac{1}{1-\frac{1}{5}t}$

$= \dfrac{4}{5}t \times \left(1 + \dfrac{1}{5}t + \dfrac{1}{25}t^2 + \dfrac{1}{125}t^3 + \cdots\right)$

$= \dfrac{4}{5}t + \dfrac{4}{25}t^2 + \dfrac{4}{125}t^3 + \dfrac{4}{625}t^4 + \cdots$

$\Rightarrow \text{P}(1 \le X \le 4) = \dfrac{4}{5} + \dfrac{4}{25} + \dfrac{4}{125} + \dfrac{4}{625} = \dfrac{624}{625}$

b $\text{P}(X \ge 2) = 1 - \text{P}(X \le 1) = 1 - \left(\dfrac{4}{5} + \dfrac{4}{25}\right) = 1 - \dfrac{24}{25} = \dfrac{1}{25}$

c $G(t) = \dfrac{4t}{5-t},\ G'(t) = \dfrac{4(5-t) + 4t}{(5-t)^2}$

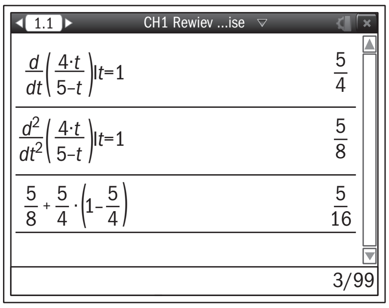
$= \dfrac{20}{(5-t)^2} \Rightarrow \text{E}(X) = G'(1) = \dfrac{20}{16} = \dfrac{5}{4}$

d $G'(t) = 20(5-t)^{-2}$,

$G''(t) = -40(5-t)^{-3} \times (-1) = 40(5-t)^{-3}$

$\Rightarrow \text{Var}(X) = G''(1) + G'(1)(1 - G'(1))$

$= 40 \times 4^{-3} + \dfrac{5}{4}\left(1 - \dfrac{5}{4}\right) = \dfrac{5}{8} - \dfrac{5}{16} = \dfrac{5}{16}$



2 a $G_X(t) = e^{0.6(t-1)},\ G_Y(t) = e^{0.12(t-1)},\ G_Z(t) = e^{0.28(t-1)}$

b $G_{X+Y+Z}(t) = G_X(t) \times G_Y(t) \times G_Z(t)$

$= e^{0.6(t-1)} \times e^{0.12(t-1)} \times e^{0.28(t-1)} = e^{t-1}$

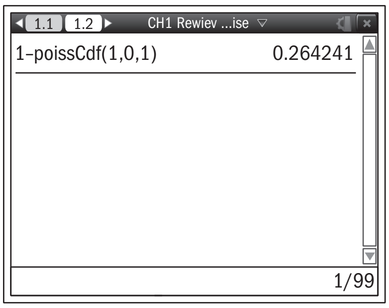
$G_{X+Y+Z}(t) = e^{t-1} \Rightarrow \text{E}(X + Y + Z)$

$= G_{X+Y+Z}(1) = e^{1-1} = e^0 = 1$

The expected number of animals at the meadow at the given moment is 1.

c $G_{X+Y+Z}(t) = e^{t-1} \Rightarrow X + Y + Z \sim \text{Po}(1)$

$\text{P}(X + Y + Z \ge 2) = 1 - \text{P}(X + Y + Z \le 1) = 0.264$



3 a i $X \sim \text{Geo}(0.75) \Rightarrow \text{P}(X = 4) = 0.0117$

ii $Y \sim \text{NB}(2, 0.75) \Rightarrow \text{P}(Y = 4)$

$$= \binom{4-1}{2-1} 0.25^2 \times 0.75^2 = \frac{27}{256} = 0.105$$

iii $\text{P}(X \geq 3) = 1 - \text{P}(X \leq 2) = 0.0625$

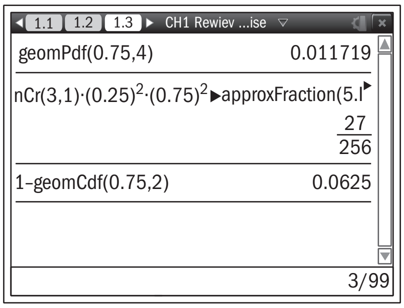
iv $Z \sim \text{NB}(6, 0.75)$

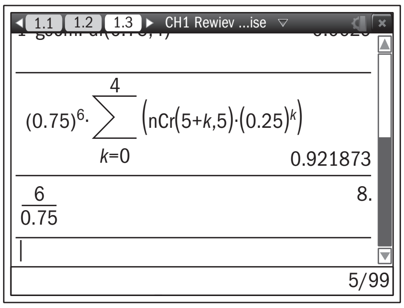
$$\Rightarrow \text{P}(Z \leq 10) = \binom{6-1}{6-1} 0.75^6$$

$$+ \binom{7-1}{6-1} 0.75^6 \times 0.25 + \ldots$$

$$+ \binom{10-1}{6-1} 0.75^6 \times 0.25^4 = 0.922$$

b $\text{E}(Z) = \dfrac{6}{0.75} = 8$, the expected number of students needed to be selected if we need six involved in the programme is eight.





4 a $F(2) = 1 \Rightarrow \dfrac{2^2}{a^2} = 1 \Rightarrow a^2 \Rightarrow 2^2 \Rightarrow a = 2$ or

~~$a = -2$~~ since a must be positive.

b $F'(x) = f(x) \Rightarrow f(x) = \begin{cases} \dfrac{2x}{2^2}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

$$= \begin{cases} \dfrac{x}{2}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

c Since $f(x)$ is increasing on the whole interval $[0, 2]$ the modal value of the variable X is 2.

5 a $G(1) = 1 \Rightarrow \dfrac{1}{a-b} = 1 \Rightarrow a - b = 1 \Rightarrow b = a - 1$

b $G'(t) = \dfrac{a - bt - t \times (-b)}{a - bt^2} = \dfrac{a}{(a - bt)^2}$

$$\Rightarrow \text{E}(X) = G'(1) = \frac{a}{(a-b)^2} = \frac{a}{1^2} = a$$

c $G'(t) = a(a - bt)^{-2}$,

$$G''(t) = a \times (-2)(a - bt)^{-3} \times (-b) = \frac{2ab}{(a - bt)^3}$$

$$\text{Var}(X) = G''(1) + G'(1)(1 - G'(1))$$

$$= \frac{2a(a-1)}{1^3} + a(1 - a)$$

$$= 2a^2 - 2a + a - a^2 = a^2 - a$$

6 a $\left.\begin{matrix} X_1 \sim \text{Po}(m),\ G_{X_1}(t) = e^{m(t-1)} \\ X_2 \sim \text{Po}(m),\ G_{X_2}(t) = e^{m(t-1)} \end{matrix}\right\} \Rightarrow G_{X_1+X_2}(t)$

$$= G_{X_1}(t) \times G_{X_2}(t) = e^{m(t-1)} \times e^{m(t-1)} = e^{m(t-1)+m(t-1)}$$

$$= e^{2m(t-1)} \Rightarrow X_1 + X_2 \sim \text{Po}(2m)$$

b 1st Base

For $n = 1$, the statement is trivial: $X_1 \sim \text{Po}(m)$

For $n = 2$, in part **a** we proved that $X_1 + X_2 \sim \text{Po}(2m)$

2nd Assumption

Let's assume that for $n = k$ this statement is true:

$$\sum_{i=1}^{k} X_1 \sim \text{Po}(km)$$

3rd Step

For $n = k + 1$, $\displaystyle\sum_{i=1}^{k+1} X_1 = \sum_{i=1}^{k} X_i + X_{k+1}$, by the assumption the first term has a Poisson distribution with the coefficient km and by the given proposition in the question $X_{k+1} \sim \text{Po}(m)$. Now we use the fact that the coefficient of the sum of two Poisson variables is the sum of the coefficients of those two variables.

$$\left.\begin{matrix} \displaystyle\sum_{i=1}^{k} X_i \sim \text{Po}(km) \\ X_{k+1} \sim \text{Po}(m) \end{matrix}\right\} \Rightarrow \sum_{i=1}^{k} X_i + X_{k+1} \sim \text{Po}(km + m)$$

$$\Rightarrow \sum_{i=1}^{k+1} X_i \sim \text{Po}\big((k+1)m\big)$$

4th Conclusion

Now we can conclude that $Y = \displaystyle\sum_{k=1}^{n} X_i \sim \text{B}(nm)$

for all the positive integer values of n.

2 Expectation algebra and Central Limit Theorem

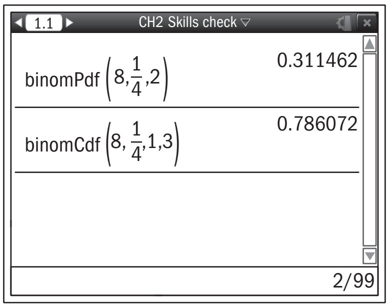
Skills check

1 $\left.\begin{aligned} np &= 2 \\ npq &= \frac{3}{2} \end{aligned}\right\} \Rightarrow 2q = \frac{3}{2} \Rightarrow q = \frac{3}{4},$

$p = \frac{1}{4},\ n = 8 \Rightarrow X \sim \mathrm{B}\left(8, \frac{1}{4}\right)$

a $\mathrm{P}(X = 2) = \binom{8}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 = 0.311$

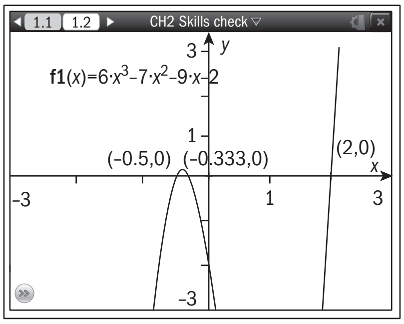
b $\mathrm{P}(1 \le X \le 3) = \sum_{k=1}^{3}\binom{8}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{8-k} = 0.786$



2 $X \sim \mathrm{Po}(m) \Rightarrow \mathrm{E}(X) = \mathrm{Var}(X)$

$7a^2 = 6a^3 - 9a - 2 \Rightarrow 6a^3 - 7a^2 - 9a - 2 = 0$

$\Rightarrow (2a + 1)(3a + 1)(a - 2) = 0$



$\Rightarrow a_1 = -\frac{1}{2},\ a_2 = -\frac{1}{3},\ a_3 = 2$

We see that all three values of a give positive values of the parameter m:

$m_1 = \frac{7}{4},\ m_2 = \frac{7}{9},\ m_3 = 28$

Exercise 2A

1 $\mathrm{E}(X) = 5.3,\ \mathrm{Var}(X) = 1.2$

a $\mathrm{E}(3X) = 3 \times 5.3 = 15.9,$
$\mathrm{Var}(3X) = 3^2 \times 1.2 = 10.8$

b $\mathrm{E}(X + 3) = 5.3 + 3 = 8.3,\ \mathrm{Var}(X + 3) = 1.2$

c $\mathrm{E}(4X + 1) = 4 \times 5.3 + 1 = 22.2,$
$\mathrm{Var}(4X + 1) = 4^2 \times 1.2 = 19.6$

d $\mathrm{E}(2X - 5) = 2 \times 5.3 - 5 = 5.6,$
$\mathrm{Var}(2X - 5) = 2^2 \times 1.2 = 4.8$

e $\mathrm{E}(kX + p) = k \times 5.3 + p,$
$\mathrm{Var}(kX + p) = k^2 \times 1.2 = 1.2k^2$

2 $X \sim \mathrm{B}\left(10, \frac{2}{5}\right) \Rightarrow \mathrm{E}(X) = 10 \times \frac{2}{5} = 4,$

$\mathrm{Var}(X) = 10 \times \frac{2}{5} \times \frac{3}{5} = \frac{12}{5} = 2.4$

a $\mathrm{E}(3X + 2) = 3 \times \mathrm{E}(X) + 2 = 3 \times 4 + 2 = 14$

b $\mathrm{Var}(3X - 2) = 3^2 \times \mathrm{Var}(X) = 9 \times \frac{12}{5} = \frac{108}{5} = 21.6$

3 $Y \sim \mathrm{Geo}\left(\frac{2}{3}\right) \Rightarrow \mathrm{E}(Y) = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1.5,$

$\mathrm{Var}(Y) = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^2} = \frac{3}{4} = 0.75$

$\mathrm{E}(2Y - 1) = 2 \times \mathrm{E}(Y) - 1 = 2 \times \frac{3}{2} - 1 = 2,$

$\mathrm{Var}(2Y - 1) = 2^2 \times \mathrm{Var}(Y) = 4 \times \frac{3}{4} = 3$

4 $Y \sim \mathrm{Po}(2) \Rightarrow \mathrm{E}(Y) = 2,\ \mathrm{Var}(Y) = 2$

a $\mathrm{E}(3 - 2Y) = 3 - 2 \times \mathrm{E}(Y) = 3 - 2 \times 2 = -1$

b $\mathrm{Var}(3 - 2Y) = 2^2 \times \mathrm{Var}(Y) = 4 \times 2 = 8$

5 $X \sim \text{NB}\left(8, \frac{1}{3}\right) \Rightarrow \text{E}(X) = \frac{8}{\frac{1}{3}} = 24,$

$$\text{Var}(Y) = \frac{8 \times \frac{2}{3}}{\left(\frac{1}{3}\right)^2} = 48$$

a $\text{E}(2X - 3) = 2 \times \text{E}(X) - 3 = 2 \times 24 - 3 = 45$

b $\text{Var}(2X - 11) = 2^2 \times \text{Var}(X) = 4 \times 48 = 192$

6 $X \sim \text{B}(15, p) \Rightarrow \text{E}(X) = np \Rightarrow 15p = 6 \Rightarrow p = \frac{2}{5}$

$$\text{Var}(X) = npq \Rightarrow \text{Var}(X) = 6 \times \frac{3}{5} = \frac{18}{5} = 3.6$$

$$\text{Var}(5X + 3) = 5^2 \times \frac{18}{5} = 90$$

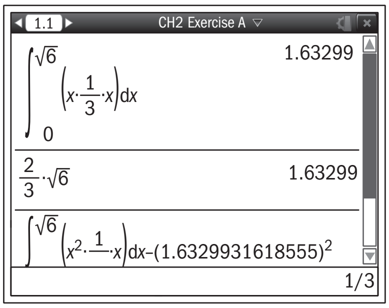
7 $\text{E}(X) = \int_0^{\sqrt{6}} x \times \frac{1}{3} x\, dx = \frac{1}{3}\int_0^{\sqrt{6}} x^2\, dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^{\sqrt{6}}$

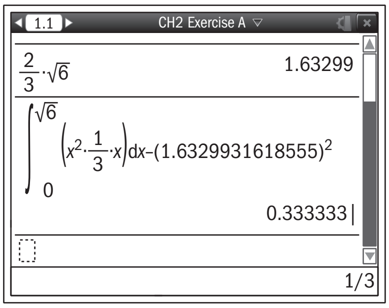
$$= \frac{1}{9} 6\sqrt{6} = \frac{2}{3}\sqrt{6}$$

$$\text{Var}(X) = \int_0^{\sqrt{6}} x^2 \times \frac{1}{3} x\, dx - \left(\frac{2}{3}\sqrt{6}\right)^2$$

$$= \frac{1}{3}\int_0^{\sqrt{6}} x^3\, dx - \frac{8}{3} = \frac{1}{3}\left[\frac{x^4}{4}\right]_0^{\sqrt{6}} - \frac{8}{3}$$

$$= \frac{1}{12} \times 6^2 - \frac{8}{3} = 3 - \frac{8}{3} = \frac{1}{3}$$





$$\text{E}(3X + 2) = 3 \times \frac{2}{3}\sqrt{6} + 2 = 2\sqrt{6} + 2,$$

$$\text{Var}(3X + 2) = 3^2 \times \frac{1}{3} = 3$$

Exercise 2B

1 **a** $\text{E}(X + Y) = 3 + (-5) = -2,$

$\text{Var}(X + Y) = 0.5 + 1.4 = 1.9$

b $\text{E}(2Y - Z) = 2 \times (-5) - 12 = -22,$

$\text{Var}(2Y - Z) = 2^2 \times 1.4 + 2.8 = 8.4$

c $\text{E}(2Z - 7X) = 2 \times 12 - 7 \times 3 = 3,$

$\text{Var}(2Z - 7X) = 2^2 \times 2.8 + 7^2 \times 0.5 = 35.7$

d $\text{E}(X - Y + Z) = 3 - (-5) + 12 = 20,$

$\text{Var}(X - Y + Z) = 0.5 + 1.4 + 2.8 = 4.7$

e $\text{E}(X + Y - Z) = 3 + (-5) - 12 = -14,$

$\text{Var}(X + Y - Z) = 0.5 + 1.4 + 2.8 = 4.7$

f $\text{E}(3Z - 2X + 4Y) = 3 \times 12 - 2 \times 3 + 4 \times (-5) = 10,$

$\text{Var}(3Z - 2X + 4Y) = 3^2 \times 2.8 + 2^2 \times 0.5$

$+ 4^2 \times 1.4 = 49.6$

2 $\text{Var}(X) = 2 \Rightarrow X \sim \text{Po}(2) \Rightarrow \text{E}(X) = 2$ and

$\text{E}(Y) = 5 \Rightarrow Y \sim \text{Po}(5) \Rightarrow \text{Var}(Y) = 5$

a $\text{E}(3X + 5Y) = 3 \times 2 + 5 \times 5 = 31$

b $\text{Var}(11Y - 7X) = 11^2 \times 5 + 7^2 \times 2 = 703$

3 $\text{E}(X) = 9 \Rightarrow n_1 p = 9 \Rightarrow \text{Var}(X) = 9(1 - p)$

$\text{E}(Y) = 4 \Rightarrow n_2 p = 4 \Rightarrow \text{Var}(X) = 4(1 - p)$

$$\text{Var}(2X - 3Y) = 2^2 \times 9(1 - p) + 3^2 \times 4(1 - p)$$

$$= 72 - 72p$$

4 $\text{E}(X) = 8 \Rightarrow \frac{r_1}{p} = 8 \Rightarrow$

$$\text{Var}(X) = 8 \times \frac{1 - p}{p} = \frac{8}{p} - 8$$

$\text{E}(Y) = 12 \Rightarrow \frac{r_2}{p} = 12 \Rightarrow$

$$\text{Var}(Y) = 12 \times \frac{1 - p}{p} = \frac{12}{p} - 12$$

$$\text{Var}(X - Y) = \frac{8}{p} - 8 + \frac{12}{p} - 12 = \frac{20}{p} - 20$$

5 Given the n independent variables and their corresponding parameters

X_n, $\text{E}(X_n)$, $\text{Var}(X_n)$, $n \in \mathbb{Z}^+$

1st Base

$n = 1$

X_1, $\text{E}(X_1)$, $\text{Var}(X_1) \Rightarrow \text{E}(aX_1) = a\text{E}(X_1),$

$\text{Var}(aX_1) = a^2\text{Var}(X_1)$

2nd Assumption

Let's assume that for $n = k$ the statement is true so the parameters of $\sum_{i=1}^{k} a_i X_i$ are calculated as follows:

$$\mathrm{E}\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i \mathrm{E}(X_i) \text{ and}$$

$$\mathrm{Var}\left(\sum_{i=1}^{k} a_i X_i\right) = \sum_{i=1}^{k} a_i^2 \mathrm{Var}(X_i)$$

3rd Step

For $n = k + 1$, $\sum_{i=1}^{k+1} X_i = \sum_{i=1}^{k} X_i + X_{k+1}$,

$$\mathrm{E}\left(\sum_{i=1}^{k+1} a_i X_i\right) = \mathrm{E}\left(\sum_{i=1}^{k} a_i X_i + a_{k+1} X_{k+1}\right)$$

$$= \mathrm{E}\left(\sum_{i=1}^{k} a_i X_i\right) + \mathrm{E}(a_{k+1} X_{k+1})$$

$$= \sum_{i=1}^{k} a_i \mathrm{E}(X_i) + a_{k+1}\mathrm{E}(X_{k+1}) = \sum_{i=1}^{k+1} a_i \mathrm{E}(X_i)$$

$$\mathrm{Var}\left(\sum_{i=1}^{k+1} a_i X_i\right) = \mathrm{Var}\left(\sum_{i=1}^{k} a_i X_i + a_{k+1} X_{k+1}\right)$$

$$= \sum_{i=1}^{k} a_i^2 \mathrm{Var}(X_i) = a_{k+1}^2 \mathrm{Var}(X_{k+1})$$

$$= \mathrm{Var}\left(\sum_{i=1}^{k+1} a_i X_i\right)$$

4th Conclusion

The parameters of a linear combination of n random variables can be calculated by:

$$\mathrm{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \mathrm{E}(X_i),$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$$

Exercise 2C

1 Let's independently flip 6 unbiased coins and record the number of heads obtained.

a

$X = x$	0	1
$P\{X = x\}$	$\frac{1}{2}$	$\frac{1}{2}$

$\mathrm{E}(X) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$,

$\mathrm{Var}(X) = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

b Since we have 6 independent flips of the same coin we add six instances of the variable X.

c Find the expected value and variance of the random variable Y.

$\mathrm{E}(Y) = 6 \times \frac{1}{2} = 3$, $\mathrm{Var}(Y) = 6 \times \frac{1}{4} = \frac{3}{2}$

d $\mathrm{E}(Y) \pm 3 \times \sqrt{\mathrm{Var}(Y)} = 3 \pm 3 \times \sqrt{\frac{3}{2}}$

$= 3 \pm 3.67 \Rightarrow y \in [-0.67, 6.67]$

We notice that all the possible outcomes $\{0, 1, 2, 3, 4, 5, 6\}$ are covered by the 99.7% empirical rule.

2 Let's roll an unbiased die 4 times. Record the number of multiples of 3 obtained.

a

$X = x$	0	1
$P\{X = x\}$	$\frac{2}{3}$	$\frac{1}{3}$

$\mathrm{E}(X) = 0 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{1}{3}$,

$\mathrm{Var}(X) = 0^2 \times \frac{2}{3} + 1^2 \times \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$

b Since we have 4 independent rolls of the same die we add four instances of the variable X.

c $\mathrm{E}(Y) = 4 \times \frac{1}{3} = \frac{4}{3}$, $\mathrm{Var}(Y) = 4 \times \frac{2}{9} = \frac{8}{9}$

3 **a** $\mathrm{E}(X + X + X) = 3 \times \mathrm{E}(X) = 3 \times 3 = 9$,

$\mathrm{Var}(X + X + X) = 3 \times \mathrm{Var}(X) = 3 \times 4 = 12$

b $\mathrm{E}(3X) = 3 \times \mathrm{E}(X) = 3 \times 3 = 9$,

$\mathrm{Var}(3X) = 3^2 \times \mathrm{Var}(X) = 9 \times 4 = 36$

c $\mathrm{Var}(X + X + X + 3X) = 3 \times \mathrm{Var}(X)$

$+ 3^2 \times \mathrm{Var}(X) = 12\ \mathrm{Var}(X) = 12 \times 4 = 48$

$\mathrm{Var}(6X) = 6^2\,\mathrm{Var}(X) = 36 \times 4 = 144 \Rightarrow$

$\mathrm{Var}(X + X + X + 3X) \neq \mathrm{Var}(6X)$

4 **a** $\mathrm{E}(X + X + X + X + X) = 5 \times \mathrm{E}(X) = 5 \times 2 = 10$

$\mathrm{Var}(X + X + X + X + X) = 5 \times \mathrm{Var}(X) = 5 \times 1 = 5$

b $\mathrm{E}(Y + Y + Y) = 3 \times \mathrm{E}(Y) = 3 \times 5 = 15$

$\mathrm{Var}(Y + Y + Y) = 3 \times \mathrm{Var}(Y) = 3 \times 3 = 9$

c $\mathrm{E}(X + X + X + X + X + Y + Y + Y)$

$= \mathrm{E}(X + X + X + X + X) + \mathrm{E}(Y + Y + Y)$

$= 10 + 15 = 25$

$\mathrm{Var}(X + X + X + X + X + Y + Y + Y)$

$= \mathrm{Var}(X + X + X + X + X) + \mathrm{Var}(Y + Y + Y)$

$= 5 + 9 = 14$

5 **a**

$X = x_i$	1	2	3	4
$P\{X = x_i\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$\mathrm{E}(X) = \frac{1}{4} \times (1 + 2 + 3 + 4) = \frac{1}{4} \times 10 = \frac{5}{2}$

$Y = y_i$	1	2	3
$P(Y = y_i)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$\mathrm{E}(Y) = \frac{1}{2} \times 1 + \frac{1}{3} \times 2 + \frac{1}{6} \times 3 = \frac{5}{3}$

b $1: \mathrm{P}(\{1, 1\}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$2: \mathrm{P}(\{1, 2\}\ or\ \{2, 1\}) = \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{2} = \frac{5}{24}$

$3: (1, 3)\ or\ (3, 1) \rightarrow \frac{1}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{2} = \frac{1}{6}$

$4: \mathrm{P}(\{2, 2\}\ or\ \{4, 1\}) = \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{2} = \frac{5}{24}$

$6: \mathrm{P}(\{2, 3\}\ or\ \{3, 2\}) = \frac{1}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{3} = \frac{1}{8}$

$8: \mathrm{P}(\{4, 2\}) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$9: \mathrm{P}(\{3,3\}) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$

$12: \mathrm{P}(\{4, 3\}) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$

$Z = z_i$	1	2	3	4	6	8	9	12
$P(Z = z_i)$	$\frac{1}{8}$	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$

$\mathrm{E}(Z) = 1 \times \frac{1}{8} + 2 \times \frac{5}{24} + 3 \times \frac{1}{6}$

$+ 4 \times \frac{5}{24} + 6 \times \frac{1}{8} + 8 \times \frac{1}{12}$

$+ 9 \times \frac{1}{24} + 12 \times \frac{1}{24} = \frac{25}{6}$

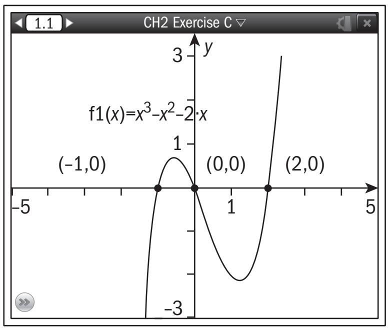
$\mathrm{E}(X) \times \mathrm{E}(Y) = \frac{5}{2} \times \frac{5}{3} = \frac{25}{6} = \mathrm{E}(Z) = \mathrm{E}(XY)$

6 $\mathrm{E}(X) \times \mathrm{E}(Y) = \mathrm{E}(XY) \Rightarrow \mu \times (\mu + 2) = \mu^3$

$\Rightarrow \mu^3 - \mu^2 - 2\mu = 0$

$\Rightarrow \mu(\mu + 1)(\mu - 2) = 0$

$\Rightarrow$ ~~$\mu = 0$~~ or ~~$\mu = -1$~~ or $\mu = 2$.



Exercise 2D

1 **a** $\mu = 1 - (-2) - 3 = 0, \sigma^2 = 0.16 + 0.25 + 1.21 = 1.62$

$Y - Z - W \sim \mathrm{N}(0, 1.62)$

$\Rightarrow \mathrm{P}(Y - Z - W < 0) = 0.5$

b $\mu = 0 + 1 + (-2) + 3 = 2,$

$\sigma^2 = 1 + 0.16 + 0.25 + 1.21 = 2.62$

$X + Y + Z + W \sim \mathrm{N}(2, 2.62)$

$\Rightarrow \mathrm{P}(X + Y + Z + W > 0) = 0.892$

c $\mu = 3 \times 0 + 1 - (-2) - 3 = 0,$

$\sigma^2 = 3^2 \times 1 + 0.16 + 0.25 + 1.21 = 10.62$

$3X + Y - Z - W \sim \mathrm{N}(0, 10.62)$

$\Rightarrow \mathrm{P}(3X + Y > Z + W)$

$= \mathrm{P}(3X + Y - Z - W > 0) = 0.5$

d $\mu = 0 - 2 \times 1 - 3 \times (-2) - 3 = -1,$

$\sigma^2 = 1 + 2^2 \times 0.16 + 3^2 \times 0.25 + 1.21 = 5.1$

$X - 2Y - 3Z - W \sim \mathrm{N}(1, 5.1)$

$\Rightarrow \mathrm{P}(X - 3Z \leq 2Y + W)$

$= \mathrm{P}(X - 2Y - 3Z - W \leq 0) = 0.329$

e $\mu = 0 - 4 \times (-2) - 3 \times 0 + 2(-2) = 4,$

$\sigma^2 = 1 + 4^2 \times 0.25 + 3^2 \times 1 + 2^2 \times 0.25 = 15$

$X - 4Z - 3X + 2Z \sim \mathrm{N}(4, 15)$

$\Rightarrow \mathrm{P}(X - 4Z \leq 3X - 2Z)$

$= \mathrm{P}(X - 4Z - 3X + 2Z \leq 0) = 0.151$

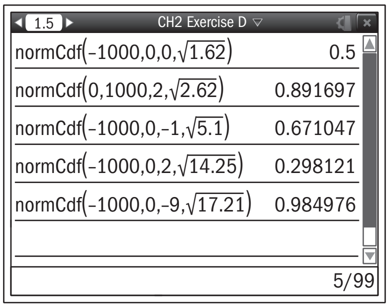
f $\mu = -3 \times 1 - 2 \times 3 = -9,$

$\sigma^2 = 1.21 + 0.16 + 2^2 \times 0.16 + 3^2 \times 1.21 = 12.9$

$W - Y - 2Y - 3W \sim \mathrm{N}(-9, 12.9)$

$\Rightarrow \mathrm{P}(W - Y \le 2Y + 3W)$

$= \mathrm{P}(W - Y - 2Y - 3W \le 0) = 0.994$



2 a $X \sim \mathrm{N}(240, 20^2), Y \sim \mathrm{N}(730, 50^2)$

$\Rightarrow Y - 3X \sim \mathrm{N}(10, 50^2 + 3^2 \times 20^2)$

$\Rightarrow \mathrm{P}(Y - 3X \ge 0) = 0.551$

b $X \sim \mathrm{N}(240, 20^2), Y \sim \mathrm{N}(730, 50^2)$

$\Rightarrow 2Y + 4X \sim \mathrm{N}(2420, 2 \times 50^2 \times 4 \times 20^2)$

$\Rightarrow \mathrm{P}(2Y + 4X \ge 2500) = 0.162$

3 a $L \sim \mathrm{N}(35, 5^2) \Rightarrow \mathrm{P}(L < 30) = 0.159$

b $V \sim \mathrm{N}(45, 8^2) \Rightarrow$

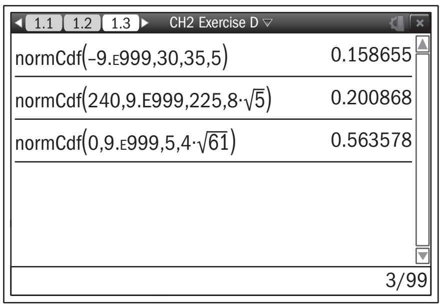
$W = +V + V + V + V \sim \mathrm{N}(5 \times 45, 5 \times 8^2)$

$= \mathrm{N}\left(225, \left(8\sqrt{5}\right)^2\right)$

$P(W > 240) = 0.201$

c $4L - 3V \sim \mathrm{N}(4 \times 35 - 3 \times 45, 4^2 \times 5^2 + 3^2 \times 8^2)$

$= \mathrm{N}\left(5, \left(4\sqrt{61}\right)^2\right) \Rightarrow \mathrm{P}(4L - 3L > 0) = 0.564$



4 a $X \sim \mathrm{N}(225, 12^2) \Rightarrow \mathrm{P}(X > 250) = 0.0186$

b $X + X + X + X \sim \mathrm{N}(4 \times 225, 4 \times 12^2) = \mathrm{N}(900, 24^2)$

$\Rightarrow \mathrm{P}(X + X + X + X > 1000) = 0.0000155$

c $Y \sim \mathrm{N}(60, 2^2) \Rightarrow 4Y \sim \mathrm{N}(4 \times 60, 4^2 \times 2^2)$

$= \mathrm{N}(240, 8^2)$

$X - 4Y \sim \mathrm{N}(225 - 240, 12^2 + 8^2)$

$= \mathrm{N}\left(-15, \left(4\sqrt{13}\right)^2\right)$

$\Rightarrow \mathrm{P}\left(X - 4Y > 0\right) = 0.149$

d $Y \sim \mathrm{N}\left(60, 2^2\right) \Rightarrow$

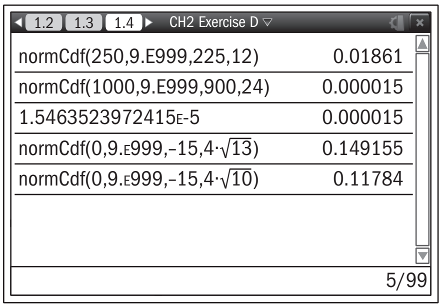
$Y + Y + Y + Y \sim \mathrm{N}\left(4 \times 60, 4 \times 2^2\right)$

$= \mathrm{N}(240, 4^2)$

$X - Y - Y - Y - Y \sim \mathrm{N}\left(225 - 240, 12^2 + 4^2\right)$

$= \mathrm{N}\left(-15, \left(4\sqrt{10}\right)^2\right)$

$\Rightarrow \mathrm{P}\left(X - Y - Y - Y - Y > 0\right) = 0.118$



Exercise 2E

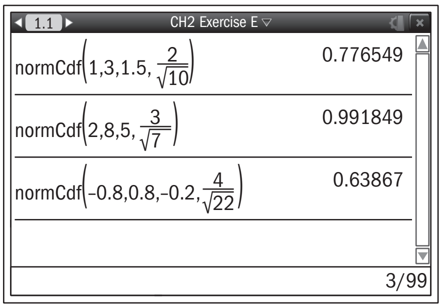
1 a $\bar{X} \sim \mathrm{N}\left(1.5, \left(\frac{2}{\sqrt{10}}\right)^2\right) \Rightarrow \mathrm{P}\left(1 \le \bar{X} \le 3\right) = 0.777$

b $\bar{X} \sim \mathrm{N}\left(5, \left(\frac{3}{\sqrt{7}}\right)^2\right) \Rightarrow \mathrm{P}\left(\left|\bar{X} - 5\right| \le 3\right)$

$= \mathrm{P}\left(2 \le \bar{X} \le 8\right) = 0.992$

c $\bar{X} \sim \mathrm{N}\left(-0.2, \left(\frac{4}{\sqrt{22}}\right)^2\right) \Rightarrow \mathrm{P}\left(\left|\bar{X}\right| \ge 0.8\right)$

$= 1 - \mathrm{P}\left(-0.8 \le \bar{X} \le 0.8\right) = 0.639$



2 $X \sim \mathrm{N}(5.5, 1.2^2) \Rightarrow \bar{X} \sim \mathrm{N}\left(5.5, \left(\frac{1.2}{\sqrt{15}}\right)^2\right)$

$\Rightarrow \mathrm{P}\left(\bar{X} < 5\right) = 0.0533$

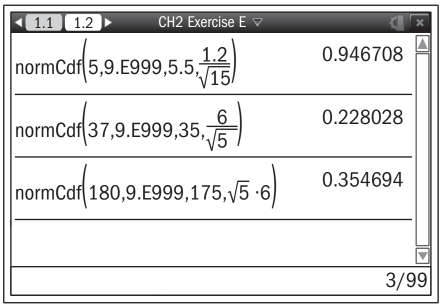
3 $X \sim N(35, 6^2) \Rightarrow \bar{X} \sim N\left(35, \left(\frac{6}{\sqrt{5}}\right)^2\right)$

a $P(\bar{X} \geq 37) = 0.228$

b $X + X + X + X + X \sim N(5 \times 35, 5 \times 6^2)$

$= N\left(175, (\sqrt{5} \times 6)^2\right)$

$\Rightarrow P(X + X + X + X + X > 180) = 0.355$



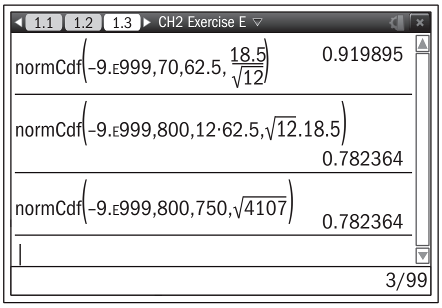
4 a $X \sim N(62.5, 18.5^2) \Rightarrow \bar{X} \sim N\left(62.5, \left(\frac{18.5}{\sqrt{12}}\right)^2\right)$

$P(\bar{X} \leq 70) = 0.920$

b $T = \underbrace{X + X + \ldots + X}_{12 \text{ terms}} \sim N(12 \times 62.5, 12 \times 18.5^2)$

$= N\left(750, (\sqrt{4107})^2\right)$

$\Rightarrow P(T \leq 800) = 0.782$



Exercise 2F

1 a $\bar{X} \sim N\left(2, \left(\frac{3}{\sqrt{30}}\right)^2\right) \Rightarrow P(1.5 \leq \bar{X} \leq 2.5) = 0.639$

b $\bar{X} \sim N\left(1.3, \left(\frac{0.2}{\sqrt{50}}\right)^2\right) \Rightarrow P(1.25 \leq \bar{X} \leq 1.35) = 0.923$

c $\bar{X} \sim N\left(-0.5, \left(\frac{1}{10}\right)^2\right) \Rightarrow P(\bar{X} \geq -0.48) = 0.421$

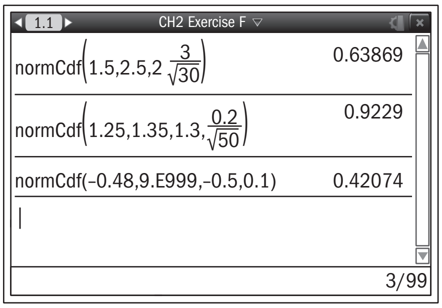
d $\bar{X} \sim N\left(400, \left(\sqrt{\frac{234}{85}}\right)^2\right) \Rightarrow P(\bar{X} < 397) = 0.0353$

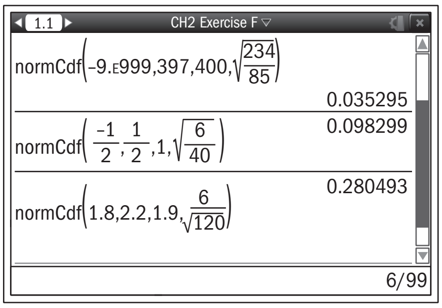
e $\bar{X} \sim N\left(1, \left(\sqrt{\frac{6}{40}}\right)^2\right) \Rightarrow P\left(|\bar{X}| < \frac{1}{2}\right)$

$= P\left(-\frac{1}{2} < \bar{X} < \frac{1}{2}\right) = 0.0983$

f $\bar{X} \sim N\left(1.9, \left(\frac{6}{\sqrt{120}}\right)^2\right) \Rightarrow P\left(|\bar{X} - 2| \geq \frac{1}{5}\right)$

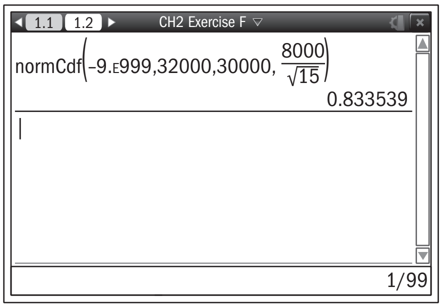
$= P\left(1.8 < \bar{X} < 2.2\right) = 0.280$





2 $\bar{X} \sim N\left(30\,000, \left(\frac{8000}{\sqrt{15}}\right)^2\right)$

$P(\bar{X} \leq 32\,000) = 0.834$

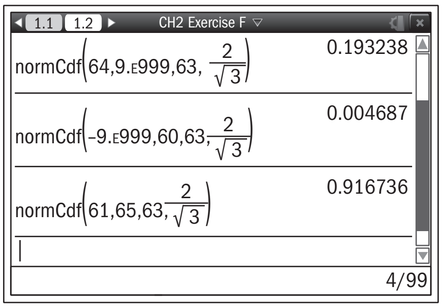


3 $X \sim N(63, 2^2) \Rightarrow \bar{X} \sim N\left(63, \left(\frac{2}{\sqrt{3}}\right)^2\right)$

a $P(\bar{X} > 64) = 0.193$

b $P(\bar{X} < 60) = 0.00469$

c $P(61 \le \bar{X} \le 65) = 0.917$



4 $X \sim N(6, 4^2) \Rightarrow \bar{X} \sim N\left(6, \left(\frac{4}{\sqrt{n}}\right)^2\right)$

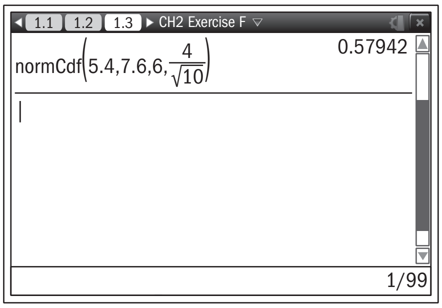
a $\bar{X} \sim N\left(6, \left(\frac{4}{\sqrt{10}}\right)^2\right) \Rightarrow P(5.4 \le \bar{X} \le 7.6) = 0.579$

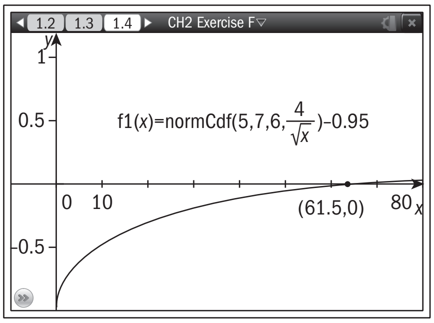
b $P(5 \le \bar{X} \le 7) = 0.95 \Rightarrow P\left(\frac{5-6}{\frac{4}{\sqrt{n}}} \le Z \le \frac{7-6}{\frac{4}{\sqrt{n}}}\right)$

$= 0.95 \Rightarrow P\left(Z \le \frac{\sqrt{n}}{4}\right) = 0.975$

$\Rightarrow \frac{\sqrt{n}}{4} = \Phi^{-1}(0.975) \Rightarrow \sqrt{n} = 4 \times 1.95996$

$\Rightarrow \sqrt{n} = 7.839856 \Rightarrow n = 61.4633 \approx 62$





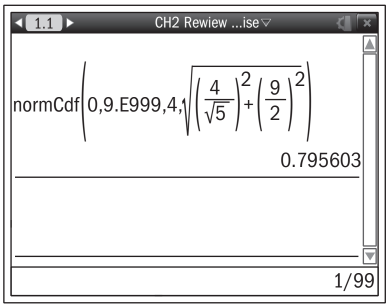
Review exercise

1 $B \sim N(202, 4^2) \Rightarrow \bar{B} \sim N\left(202, \left(\frac{4}{\sqrt{5}}\right)^2\right)$

$S \sim N(198, 9^2) \Rightarrow \bar{S} \sim N\left(198, \left(\frac{9}{\sqrt{4}}\right)^2\right)$

$\bar{B} - \bar{S} \sim N\left(4, \left(\frac{4}{\sqrt{5}}\right)^2 + \left(\frac{9}{\sqrt{4}}\right)^2\right)$

$\Rightarrow P(\bar{B} > \bar{S}) = P(\bar{B} - \bar{S} > 0) = 0.796$



2 **a** $Var(2X) = (E(X))^2 - 5 \Rightarrow 2^2 Var(X) = (E(X))^2 - 5$
$\Rightarrow 4m = m^2 - 5 \Rightarrow m^2 - 4m - 5 = 0$

$\Rightarrow (m-5)(m+1) = 0 \Rightarrow m = 5$ or ~~$m = -1$~~.
Variance is always positive.

b $P(X \ge 6) = 1 - P(X \le 5) = 0.384$

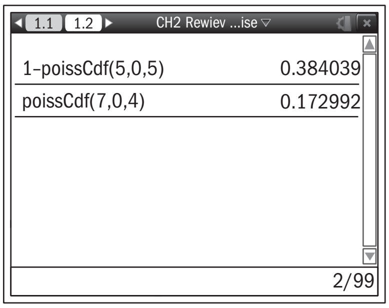
c $Var(3Y) = 18 \Rightarrow 3^2 Var(Y) = 18$

$\Rightarrow Var(Y) = 2 \Rightarrow Y \sim Po(2)$

$\Rightarrow X + Y \sim Po(5 + 2)$

$\Rightarrow X + Y \sim Po(7) \Rightarrow P(X + Y < 5)$

$= P(X + Y \le 4) = 0.173$



d $E(Z) = E(3X - 4Y) = 3 \times 5 - 4 \times 2 = 7$
$Var(Z) = Var(3X - 4Y) = 3^2 \times 5 + 4^2 \times 2 = 77$

e The random variable Z has no Poisson distribution since $7 = E(Z) \ne Var(Z) = 77$.

3 **a** $X \sim N(400, 20^2) \Rightarrow P(X > 450) = 0.00621$

b $Y \sim N(350, 15^2) \Rightarrow Y + Y \sim N(2 \times 350, 2 \times 15^2)$

$\Rightarrow P(Y + Y < 670) = 0.0787$

c $Z \sim N(320, 12^2)$

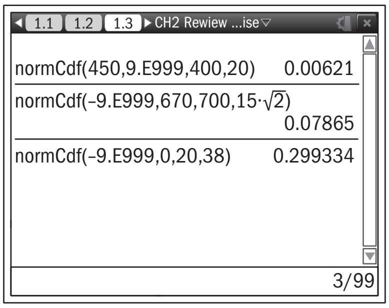
$\Rightarrow X + Z \sim N(400 + 320, 20^2 + 12^2)$

$= N\left(720, (4\sqrt{34})^2\right)$

d $X + Z - 2Y \sim \mathrm{N}\left(720 - 2\times 350, \left(4\sqrt{34}\right)^2 + 2^2\times 15^2\right)$

$= \mathrm{N}\left(20, 38^2\right)$

$\Rightarrow \mathrm{P}(X + Z - 2Y < 0) = 0.299$



4 a $X \sim \mathrm{B}(n, p) \Rightarrow \mathrm{Var}(X) = 6 \Rightarrow npq = 6$

$n = 27 \Rightarrow 27\times pq = 6 \Rightarrow p(1-p) = \frac{2}{9}$

$\Rightarrow 9p^2 - 9p + 2 = 0 \Rightarrow (3p-1)(3p-2) = 0$

$p = \frac{1}{3}$ or $p = \frac{2}{3}$

$\mathrm{E}(3X - 7) = 3\mathrm{E}(X) - 7 = 3np - 7$

$$\Rightarrow \begin{cases} \mathrm{E}(3X-7) = 3\times 27\times \frac{1}{3} - 7 = 20 \\ \mathrm{E}(3X-7) = 3\times 27\times \frac{2}{3} - 7 = 47 \end{cases}$$

b $Z \sim \mathrm{Po}(m) \Rightarrow (\mathrm{Var}(Z))^2 = \mathrm{E}(Z) + 12$

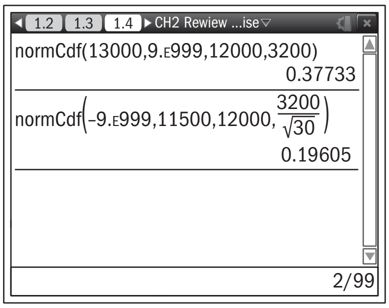
$\Rightarrow m^2 - m - 12 = 0 \Rightarrow (m-4)(m+3) = 0$

$m = 4$ *or* ~~$m = -3$~~ Variance is always positive.

$\mathrm{Var}(5 + 2Z) = 2^2\,\mathrm{Var}(Z) = 4\times 4 = 16$

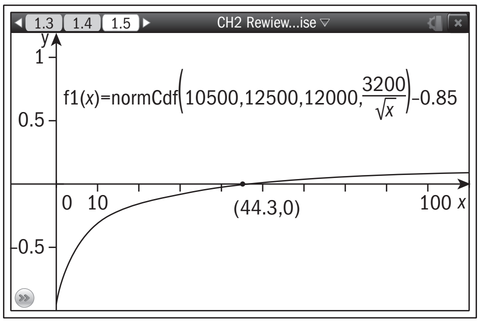
5 a $X \sim \mathrm{N}\left(12\,000, 3200^2\right) \Rightarrow \mathrm{P}(X > 13\,000) = 0.377$

b $\bar{X} \sim \mathrm{N}\left(12\,000, \left(\frac{3200}{\sqrt{30}}\right)^2\right) \Rightarrow \mathrm{P}(\bar{X} < 11500) = 0.196$



c $\bar{X} \sim \mathrm{N}\left(12\,000, \left(\frac{3200}{\sqrt{n}}\right)^2\right)$

$\Rightarrow \mathrm{P}(1050 < \bar{X} < 12\,500) = 0.85 \Rightarrow n = 44.3 \approx 45$



6 a $\mathrm{E}(X - \mathrm{E}(X))^2 = \mathrm{E}\left(X^2 - 2X\,\mathrm{E}(X) + (\mathrm{E}(X))^2\right)$

$= \mathrm{E}\left(X^2\right) - 2\mathrm{E}(X)\times\mathrm{E}(X) + (\mathrm{E}(X))^2$

$= \mathrm{E}\left(X^2\right) - 2(\mathrm{E}(X))^2 + (\mathrm{E}(X))^2$

$= \mathrm{E}\left(X^2\right) - \underbrace{(\mathrm{E}(X))^2}_{\geq 0} \geq \mathrm{E}\left(X^2\right)$

b $X \sim \mathrm{Geo}(p)$ and $Y \sim \mathrm{Geo}(q)$, where $p + q = 1$

$\Rightarrow \mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y) = \frac{1}{p} + \frac{1}{q}$

$= \frac{q+p}{pq} = \frac{1}{pq}$

$\Rightarrow \mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

$= \frac{q}{p^2} + \frac{p}{q^2} = \frac{q^3 + p^3}{p^2q^2}$

$= \frac{\overbrace{(q+p)}^{1}(q^2 - pq + p^2)}{p^2q^2}$

$= \frac{(q+p)^2 - 3pq}{p^2q^2} = \frac{1 - 3pq}{p^2q^2}$

$\mathrm{E}(X + Y)(\mathrm{E}(X + Y) - 3)$

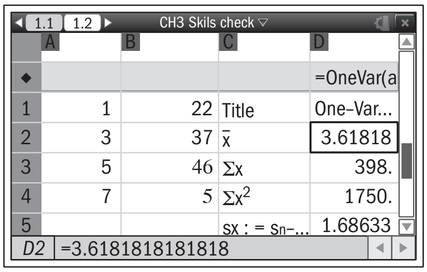
$= \frac{1}{pq}\left(\frac{1}{pq} - 3\right) = \frac{1}{p^2q^2} - \frac{3}{pq}$

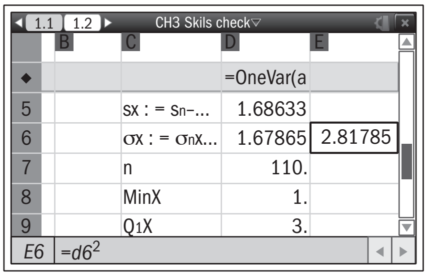
$= \frac{1 - 3pq}{p^2q^2} = \mathrm{Var}(X + Y)$ **Q.E.D.**

3 Exploring statistical analysis methods

Skills check

1





$\bar{x} = 3.62,\ \sigma^2 = 1.67865^2 = 2.82$

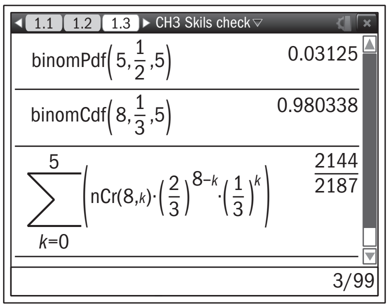
2 a $X \sim \mathrm{B}\left(5, \frac{1}{2}\right) \Rightarrow \mathrm{P}(X = 5)$

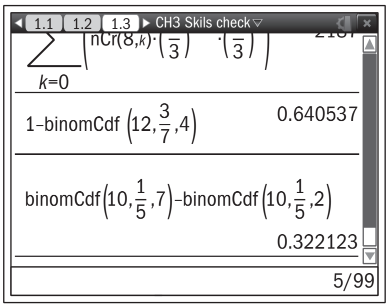
$= \binom{5}{5}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32} = 0.03125$

b $X \sim \mathrm{B}\left(10, \frac{1}{5}\right) \Rightarrow \mathrm{P}(3 \le X < 8)$

$= \mathrm{P}(3 \le X \le 7) = \sum_{k=3}^{7}\binom{10}{k}\left(\frac{4}{5}\right)^{10-k}\left(\frac{1}{5}\right)^k$

$= 0.322$

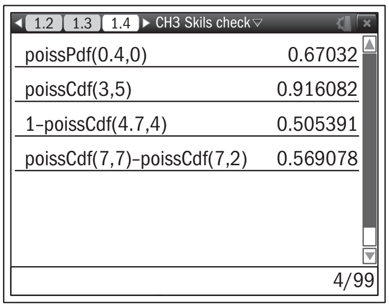




3 a $Y \sim \mathrm{Po}(0.4) \Rightarrow \mathrm{P}(Y = 0) = 0.670$

b $Y \sim \mathrm{Po}(7) \Rightarrow \mathrm{P}(3 \le X < 8)$

$= \mathrm{P}(X \le 7) - \mathrm{P}(X \le 2) = 0.569$

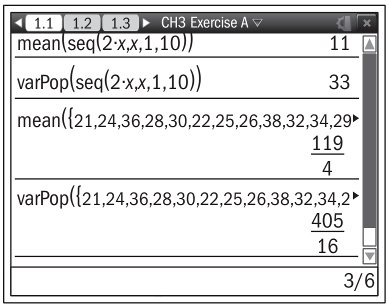


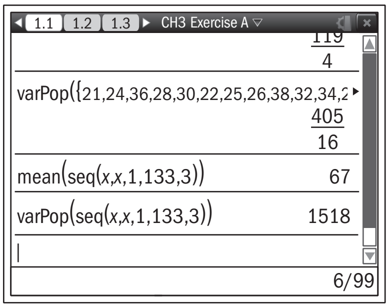
Exercise 3A

1 a $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \Rightarrow \bar{x} = 11,\ s^2 = 33$

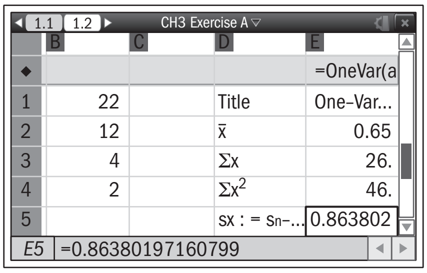
b $\{21, 24, 36, 28, 30, 22, 25, 26, 38, 32, 34, 29, 37, 33, 32, 34, 29, 37, 33, 31, 30\} \Rightarrow \bar{x} = 29.75,\ s^2 = 25.3$

c $\{1, 4, 7, 10, \ldots, 133\} \Rightarrow \bar{x} = 67,\ s^2 = 1518$



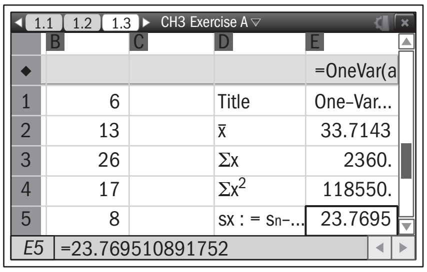


2 $\bar{x} = 0.65,\ s = 0.864$



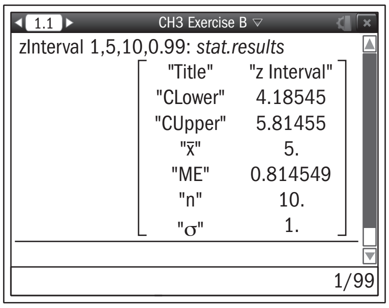
3 **a** $\bar{x} = 33.7$

b $s = 23.8$

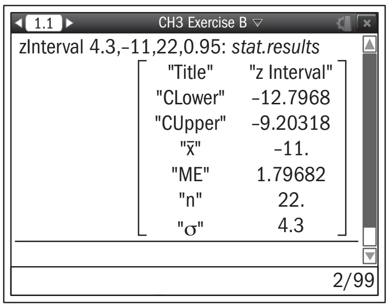


Exercise 3B

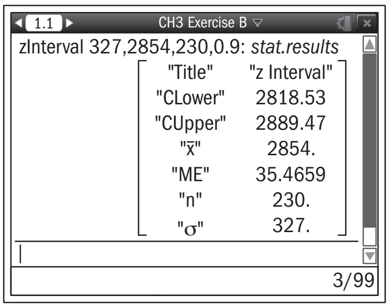
1 **a** 99% CI [4.19, 5.81]



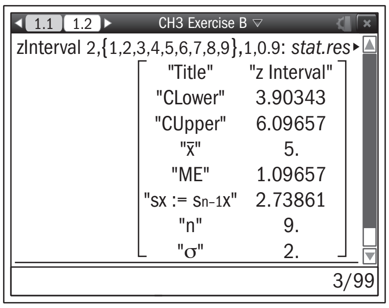
b 95% CI [–12.8, –9.20]



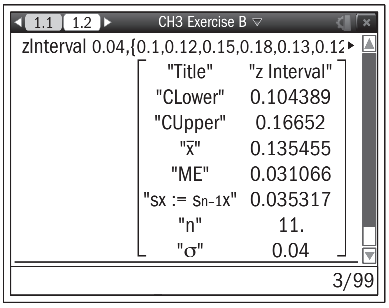
c 90% CI [2819, 2889]



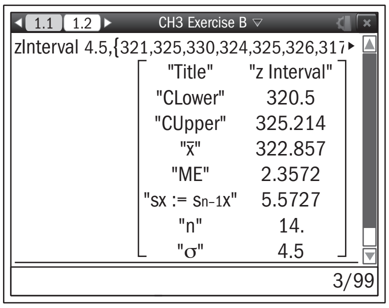
2 **a** 90% CI [3.90, 6.10]



b 99% CI [0.104, 0.167]



c 95% CI [320, 325]



3 $|\bar{x} - \mu| \le 2 \Rightarrow |\mu - \bar{x}| \le 2 \Rightarrow -2 \le \mu - \bar{x} \le 2$

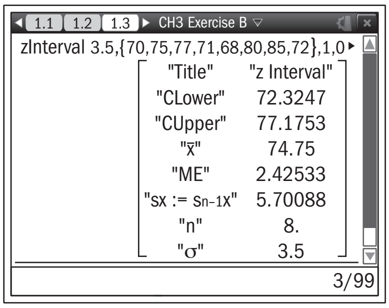
$\Rightarrow \bar{x} - 2 \le \mu \le \bar{x} + 2$

$\sigma = 5.5 \Rightarrow 1.960 \times \frac{5.5}{\sqrt{n}} = 2$

$\Rightarrow \sqrt{n} = 5.39 \Rightarrow n = 29.052$

We should take at least 30 elements.

4 95% CI for the weight if elementines is [72.3 g, 77.2 g]



5 **a** $\bar{x} = \frac{14.2 + 17.4}{2} = 15.8$

b $\sigma = 3 \Rightarrow 1.645 \times \frac{3}{\sqrt{n}} = 15.8 - 14.2$

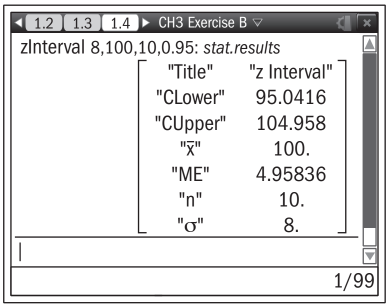
$\Rightarrow \sqrt{n} = 3.08 \Rightarrow n = 9.51$

The sample size should be 10.

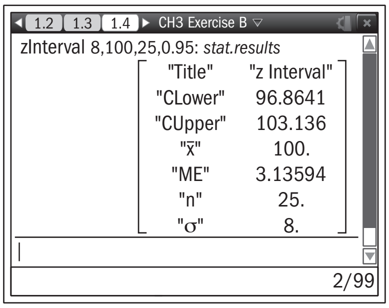
Investigation

Let's consider samples of different sizes: all have the mean value $\bar{x} = 100$ and they come from a population with $\sigma^2 = 64$.

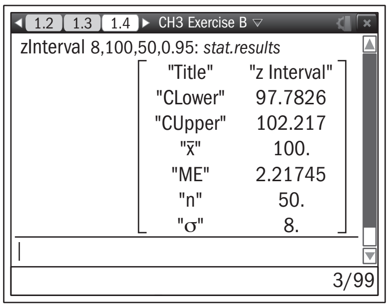
a **i** $n = 10 \Rightarrow$ 95% CI [95.0, 105.0]



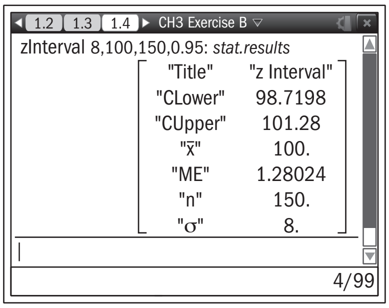
ii $n = 25 \Rightarrow$ 95% CI [96.9, 103.1]



iii $n = 50 \Rightarrow$ 95% CI [97.8, 102.2]



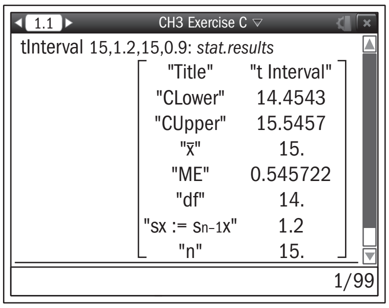
iv $n = 150 \Rightarrow$ 95% CI [98.7, 101.3]



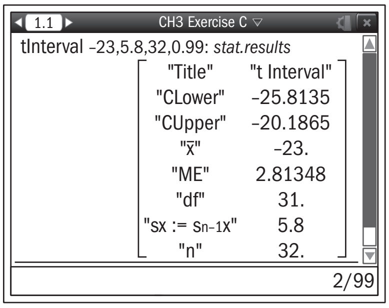
b At the same significance level the larger sample size we take the narrower a confidence interval we get. In the formula $\left[\bar{x} - \frac{\sigma}{\sqrt{n}} \times z_{\frac{\alpha}{2}}, \bar{x} + \frac{\sigma}{\sqrt{n}} \times z_{\frac{\alpha}{2}}\right]$ the length of the interval is symmetrical with respect to the mean of the sample by the same term that is obtained by division by $\sqrt{n}$ therefore the larger n we take the smaller number we obtain.

Exercise 3C

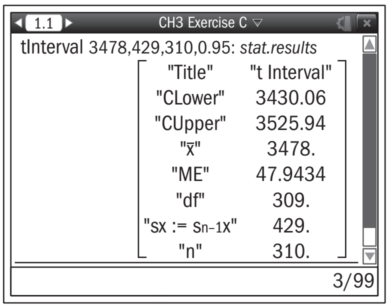
1 **a** 90% CI [14.45, 15.55]



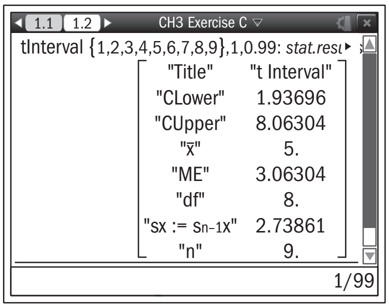
b 99% CI [−25.81, −20.19]



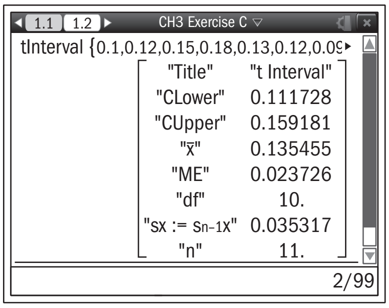
c 95% CI [3430, 3526]



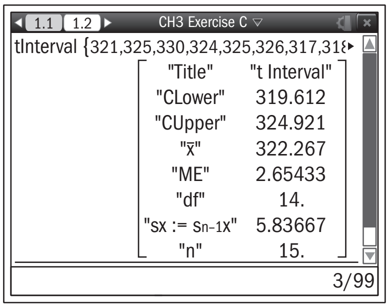
2 **a** 99% CI [1.94, 8.06]



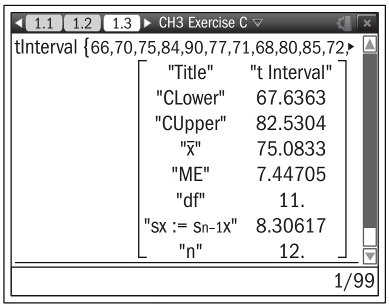
b 95% CI [0.112, 0.159]



c 90% CI [319.6, 324.9]



3 99% CI for the weight of mandarins is [67.6 g, 82.5 g]

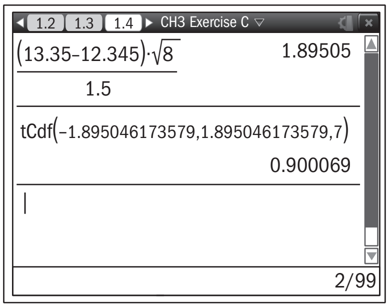


4 a $\bar{x} = \dfrac{12.345 + 14.355}{2} = 13.35$

b $s = 1.5 \Rightarrow t_c \times \dfrac{1.5}{\sqrt{8}} = 13.35 - 12.345$

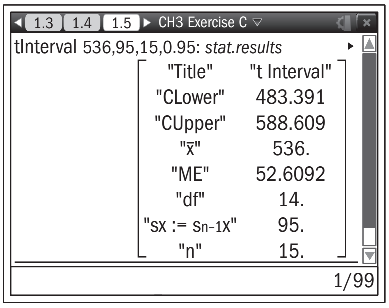
$\Rightarrow t_c = 1.895$

$\Rightarrow \text{P}(-1.895 < t < 1.895 \mid \nu = 7) = 0.900$

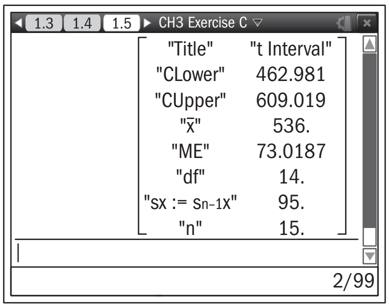


The confidence level is 90%.

5 a 95% CI [483.4, 588.6]



b 99% CI [463.0, 609.0]

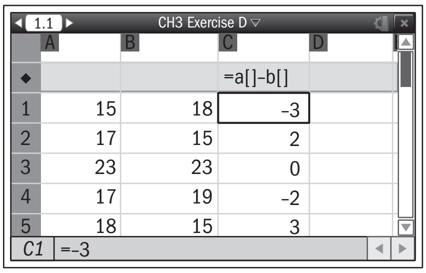


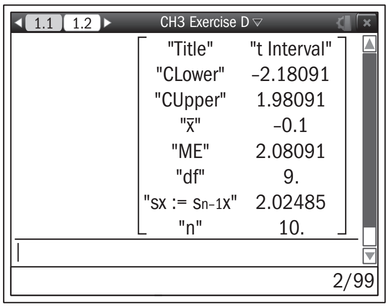
c For the same set of data, a higher significance level will mean a wider confidence interval. The result is expected since the confidence interval is symmetrical about the mean value

$\bar{x} \pm \dfrac{s}{\sqrt{n}} \times t_c$

Exercise 3D

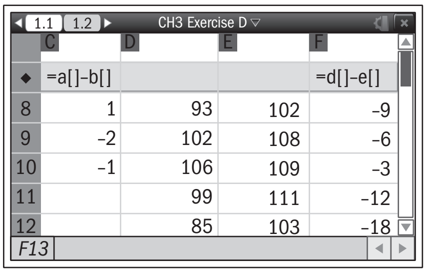
1 a

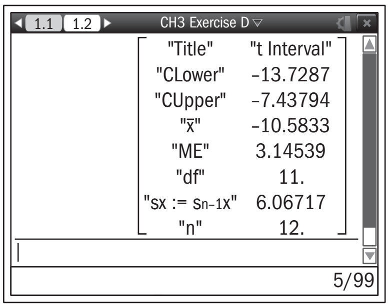




99% CI [−2.18, 1.98]

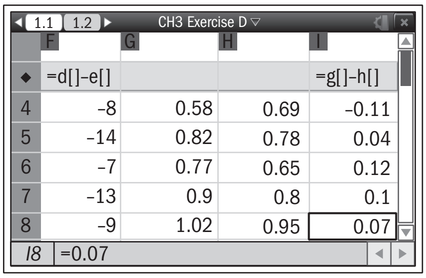
b

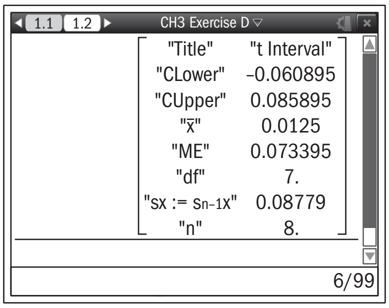




90% CI [−13.73, −7.44]

c



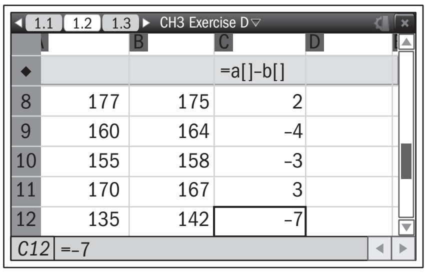


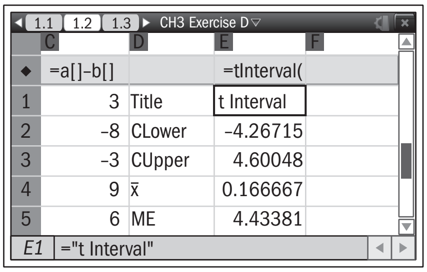
95% CI [−0.069, 0.859]

2 a

Blood sample	1	2	3	4	5	6	7	8	9	10	11	12
Bob	144	153	170	183	125	95	148	177	160	155	170	135
Rick	141	161	173	174	119	104	135	175	164	158	167	142
Bob–Rick	3	−8	−3	9	6	−9	13	2	−4	−3	3	−7

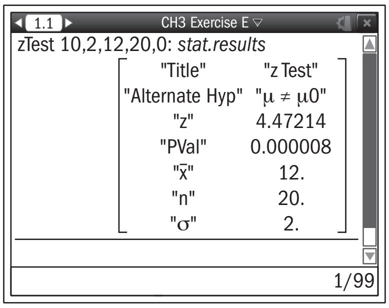
b 95% CI [−4.27, 4.60]





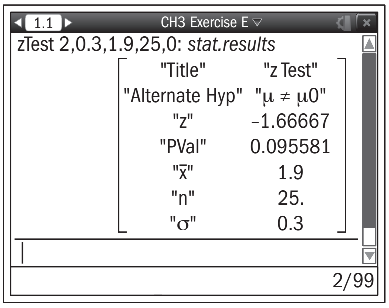
Exercise 3E

1 a $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0, \alpha = 0.1$



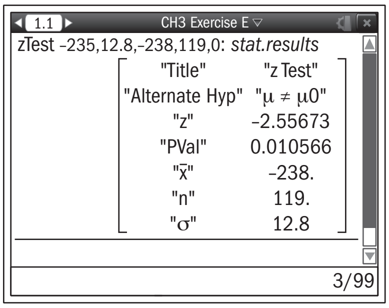
Since the p-value is 0.000008 < 0.1 we reject the null hypothesis at the 10% significance level.

b $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0, \alpha = 0.05$



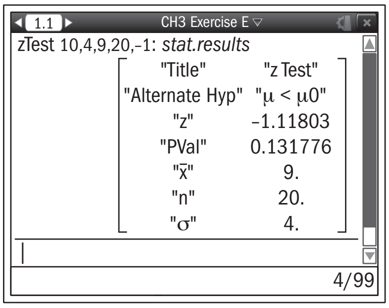
Since the p-value is 0.095581 > 0.05 we have no sufficient evidence to reject the null hypothesis at the 5% significance level.

c $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0, \alpha = 0.01$



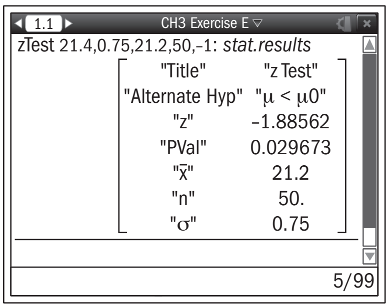
Since the p-value is 0.010566 > 0.01 we have no sufficient evidence to reject the null hypothesis at the 1% significance level.

2 a $H_0 : \mu = \mu_0, H_1 : \mu < \mu_0, \alpha = 0.1$



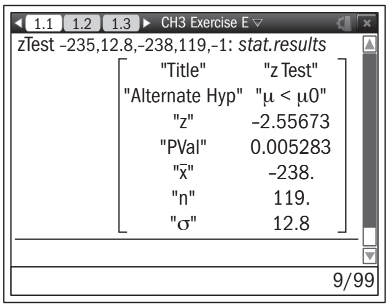
Since the p-value is 0.131776 > 0.1 we have no sufficient evidence to reject the null hypothesis at the 10% significance level.

b $H_0 : \mu = \mu_0, H_1 : \mu < \mu_0, \alpha = 0.01$



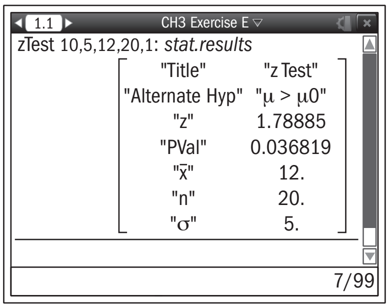
Since the p-value is 0.029673 > 0.01 we have no sufficient evidence to reject the null hypothesis at the 1% significance level.

c $H_0 : \mu = \mu_0$, $H_1 : \mu < \mu_0$, $\alpha = 0.01$



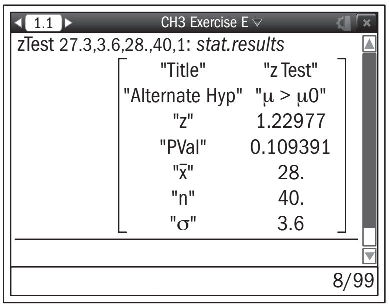
Since the p-value is 0.005283 < 0.01 we reject the null hypothesis at the 1% significance level.

3 a $H_0 : \mu = \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.05$



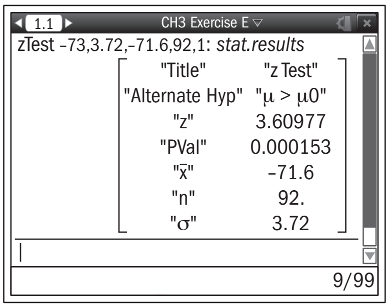
Since the p-value is 0.036819 < 0.05 we reject the null hypothesis at the 5% significance level.

b $H_0 : \mu = \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.1$



Since the p-value is 0.109391 > 0.1 we have no sufficient evidence to reject the null hypothesis at the 10% significance level.

c $H_0 : \mu = \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.01$

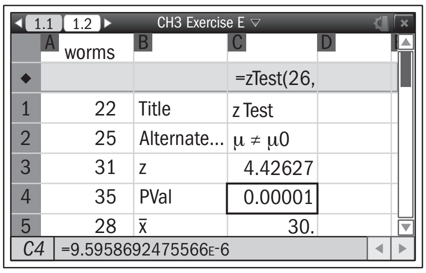


Since the p-value is 0.000153 < 0.01 we reject the null hypothesis at the 1% significance level.

4 a H_0: "The mean weight is 26 g." ($\mu = 26$)

H_1: "The mean weight is not 26 g." ($\mu \neq 26$)

b Use the z-test.

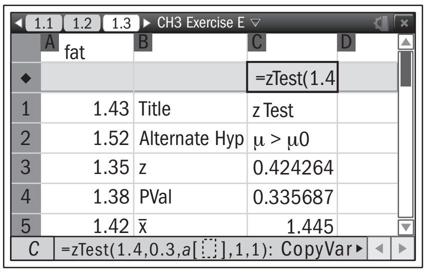


Since the p-value is 0.00001 < 0.01 we reject the null hypothesis at the 1% significance level and conclude that the harvested snails are not from the population.

5 a H_0:"The mean level of fat in the drink is 1.4 g." ($\mu = 1.4$)

H_1:"The mean level of fat in the drink is more than 1.4 g." ($\mu > 1.4$)

b Use the z-test.

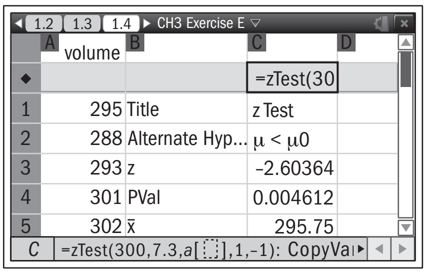


Since the p-value is 0.335687 > 0.05 we have no sufficient evidence to reject the null hypothesis at the 5% significance level and conclude that the company's claim is correct.

6 a H_0: "The mean volume of juice in the bottle is 300 ml." ($\mu = 300$)

H_1: "The mean volume of juice in the bottle is less than 300 ml."($\mu < 300$)

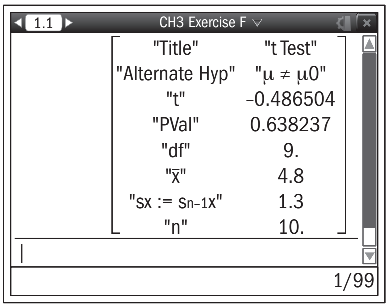
b Use the z-test.



Since the p-value is 0.004612 < 0.1 we reject the null hypothesis at the 10% significance level and conclude that the bottles contain less volume than stated.

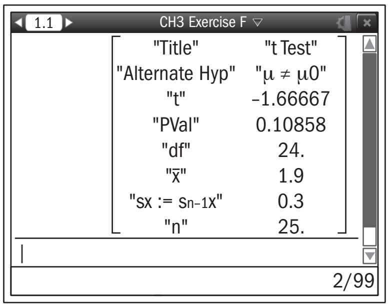
Exercise 3F

1 **a** $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$, $\alpha = 0.05$



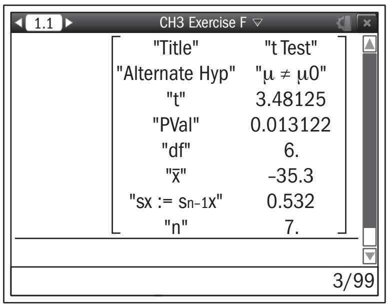
Since the *p*-value 0.6382 > 0.05 we have no sufficient evidence to reject the null hypothesis at the 5% significance level.

b $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$, $\alpha = 0.1$



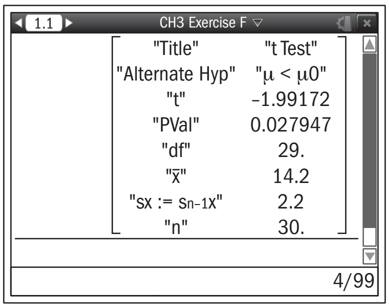
Since the *p*-value 0.10858 > 0.1 we have no sufficient evidence to reject the null hypothesis at the 10% significance level.

c $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$, $\alpha = 0.01$



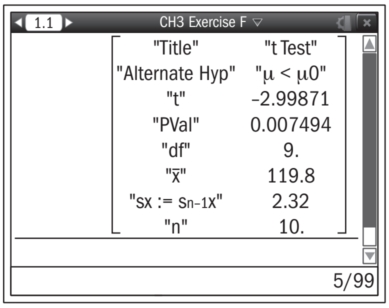
Since the *p*-value 0.013122 > 0.01 we have no sufficient evidence to reject the null hypothesis at the 1% significance level.

2 **a** $H_0 : \mu = \mu_0$, $H_1 : \mu < \mu_0$, $\alpha = 0.05$



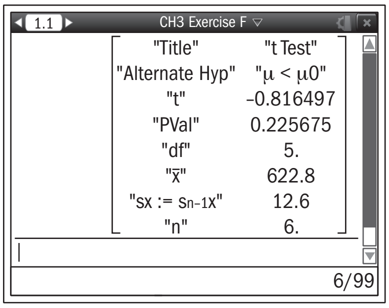
Since the *p*-value is 0.027947 < 0.05 we reject the null hypothesis at the 5% significance level.

b $H_0 : \mu = \mu_0$, $H_1 : \mu < \mu_0$, $\alpha = 0.01$



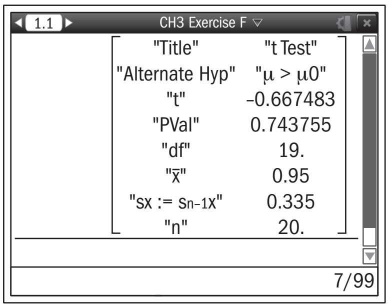
Since the *p*-value is 0.007494 < 0.01 we reject the null hypothesis at the 1% significance level.

c $H_0 : \mu = \mu_0$, $H_1 : \mu < \mu_0$, $\alpha = 0.1$



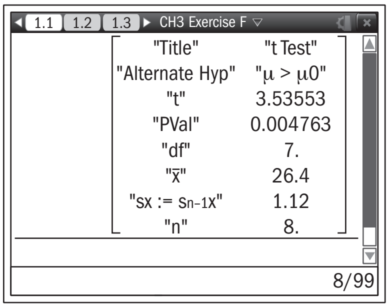
Since the *p*-value 0.225675 > 0.1 we have no sufficient evidence to reject the null hypothesis at the 10% significance level.

3 **a** $H_0 : \mu = \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.1$



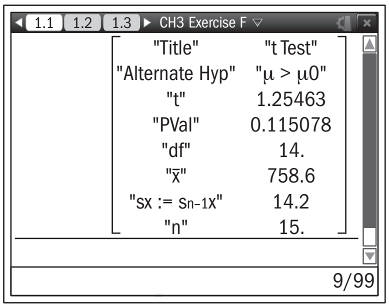
Since the *p*-value 0.743755 > 0.1 we have no sufficient evidence to reject the null hypothesis at the 10% significance level.

b $H_0 : \mu = \mu_0$, $H_1 : \mu > \mu_0$, $\alpha = 0.05$



Since the *p*-value 0.004763 < 0.05 we reject the null hypothesis at the 5% significance level.

c $H_0: \mu = \mu_0$, $H_1: \mu > \mu_0$, $\alpha = 0.01$

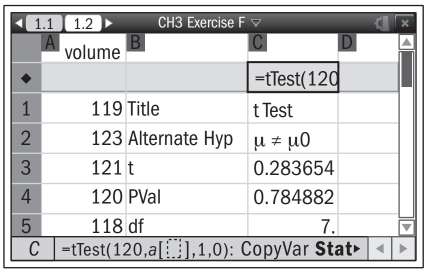


"Title"	"t Test"
"Alternate Hyp"	"μ > μ0"
"t"	1.25463
"PVal"	0.115078
"df"	14.
"x̄"	758.6
"sx := sn-1x"	14.2
"n"	15.

Since the p-value $0.115078 > 0.01$ we have no sufficient evidence to reject the null hypothesis at the 1% significance level.

4 H_0: "The mean volume is 120 ml." ($\mu = 120$)

H_1: "The mean volume is not 120 ml." ($\mu \neq 120$)



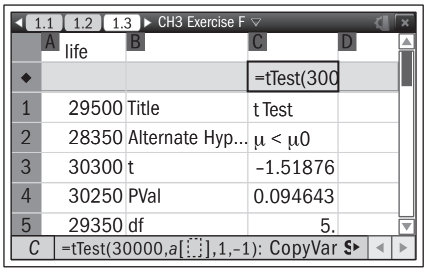
	A volume	B	C	D
◆			=tTest(120	
1	119	Title	t Test	
2	123	Alternate Hyp	μ ≠ μ0	
3	121	t	0.283654	
4	120	PVal	0.784882	
5	118	df	7.	

C =tTest(120,a[ ],1,0): CopyVar Stat

Since the p-value $0.283654 > 0.01$ we have no sufficient evidence to reject the null hypothesis at the 1% significance level and conclude that the factory advertised a correct volume of a particular ice-cream product.

5 H_0: "The mean life expectancy is 30 000 hours." ($\mu = 30\,000$)

H_1: "The mean life expectancy is less than 30 000 hours." ($\mu < 30\,000$)



	A life	B	C	D
◆			=tTest(300	
1	29500	Title	t Test	
2	28350	Alternate Hyp...	μ < μ0	
3	30300	t	-1.51876	
4	30250	PVal	0.094643	
5	29350	df	5.	

C =tTest(30000,a[ ],1,-1): CopyVar S

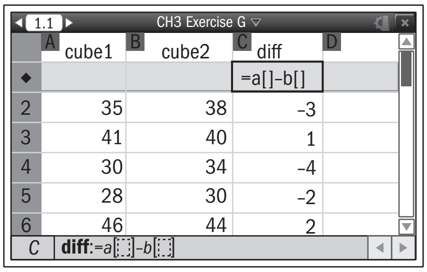
Since the p-value $0.094543 < 0.1$ we reject the null hypothesis at the 10% significance level and conclude that the manufacturer claims a longer life expectancy of the LED lamps.

Exercise 3G

1 H_0: "There is no difference in finishing times." ($\mu_d = 0$)

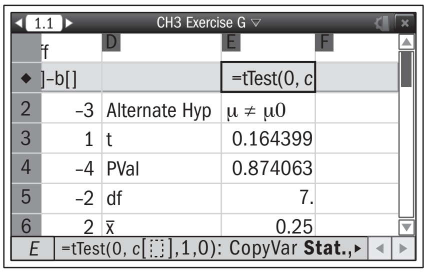
H_1: "There is a difference in finishing times." ($\mu_d \neq 0$)

Use the t-test on the difference of times on those two different cubes.



	A cube1	B cube2	C diff	D
◆			=a[]-b[]	
2	35	38	-3	
3	41	40	1	
4	30	34	-4	
5	28	30	-2	
6	46	44	2	

C diff:=a[ ]-b[ ]



	f	D	E	F
◆	]-b[]		=tTest(0, c	
2	-3	Alternate Hyp	μ ≠ μ0	
3	1	t	0.164399	
4	-4	PVal	0.874063	
5	-2	df	7.	
6	2	x̄	0.25	

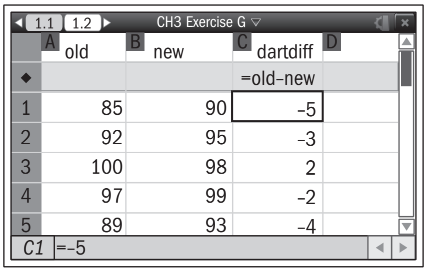
E =tTest(0, c[ ],1,0): CopyVar Stat.,

Since the p-value $0.874063 > 0.05$ we have no sufficient evidence to reject the null hypothesis at the 5% significance level and conclude that there is no difference in finishing times on the two Rubik's Cubes.

2 H_0: "There is no difference in the scores." ($\mu_d = 0$)

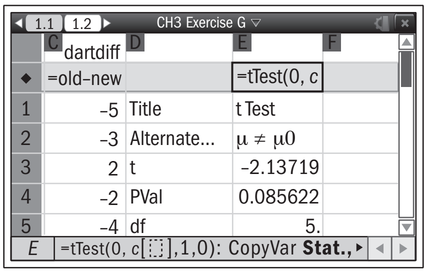
H_1: "There is a difference in the scores." ($\mu_d \neq 0$)

Use the t-test on the difference of scores on the two types of dart.



	A old	B new	C dartdiff	D
◆			=old-new	
1	85	90	-5	
2	92	95	-3	
3	100	98	2	
4	97	99	-2	
5	89	93	-4	

C1 =-5



	C dartdiff	D	E	F
◆	=old-new		=tTest(0, c	
1	-5	Title	t Test	
2	-3	Alternate...	μ ≠ μ0	
3	2	t	-2.13719	
4	-2	PVal	0.085622	
5	-4	df	5.	

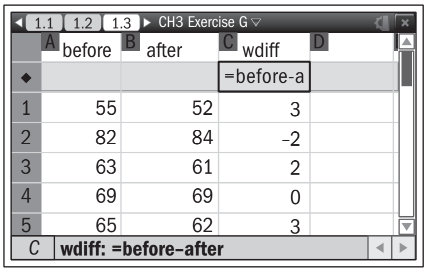
E =tTest(0, c[ ],1,0): CopyVar Stat.,

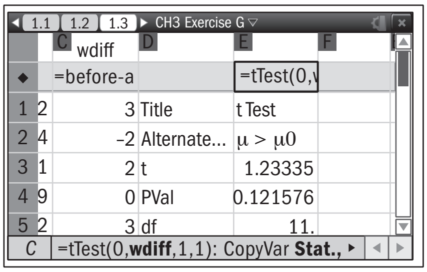
Since the p-value $0.085622 < 0.1$ we have to reject the null hypothesis at the 10% significance level and conclude that players score a better result by using the new type of dart.

3 H_0: "There is no difference in the weights." ($\mu_d = 0$)

H_1: "Students who join the programme drop some weight." ($\mu_d > 0$)

Use the t-test on the difference of weights before and after the programme.





Since the p-value is $0.121576 > 0.05$ we have no sufficient evidence to reject the null hypothesis at the 5% significance level and conclude that there is no difference in weight before and after the programme.

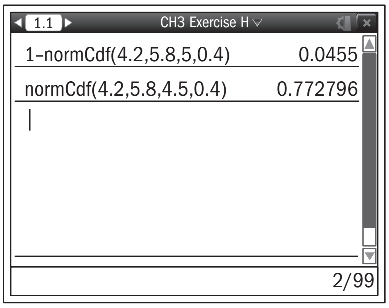
Exercise 3H

1 a $H_0 : X \sim N(5, 0.4^2)$

$\Rightarrow \alpha = 1 - P(4.2 \le X \le 5.8) = 0.0455$

b $H_1 : X \sim N(4.5, 0.4^2) \Rightarrow$

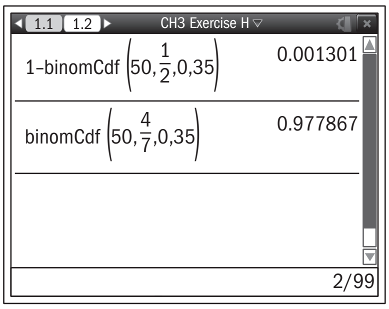
$\beta = P(4.2 \le X \le 5.8) = 0.773$



2 a $H_0 : X \sim B\left(50, \frac{1}{2}\right) \Rightarrow E(X) = 50 \times \frac{1}{2} = 25$

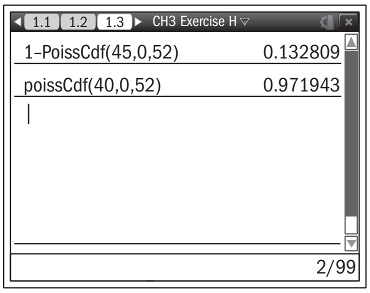
$\alpha = 1 - P(X \le 35) = 0.00130$

b $H_1 : X \sim B\left(50, \frac{4}{7}\right) \Rightarrow \beta = P(X \le 35) = 0.978$



3 a $H_0 : X \sim Po(45) \Rightarrow \alpha = 1 - P(X \le 52) = 0.133$

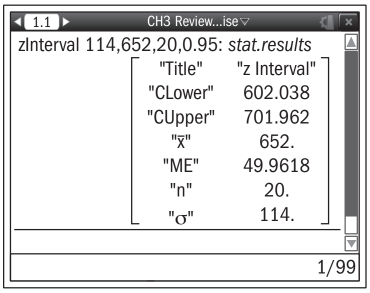
b $H_1 : X \sim Po(40) \Rightarrow \beta = P(X \le 52) = 0.972$



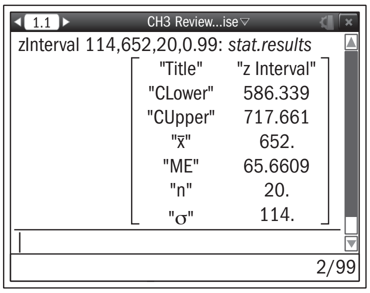
Review exercise

1 Use z-interval since the standard deviation is known and the value is 114 g.

a 95% CI [602, 702]



b 99% CI [586, 718]



c $95\% < 99\% \Rightarrow [602, 702] \subset [586, 718]$

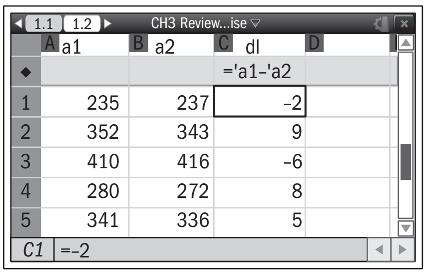
We notice that a higher significance level means a wider confidence level.

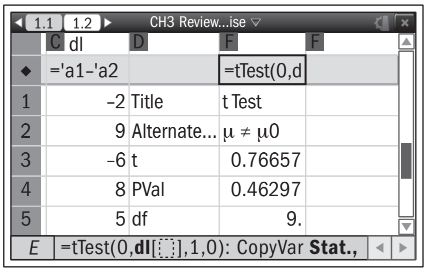
2 a

Patient	A	B	C	D	E	F	G	H	I	J
Analyzer I	235	352	410	280	341	325	428	388	272	310
Analyzer II	237	343	416	272	336	329	413	396	265	315
Difference	−2	9	−6	8	5	−4	15	−8	7	−5

b H_0: "There is no difference in potassium levels." ($\mu_d = 0$)

H_1: "There is a difference in potassium levels." ($\mu_d \ne 0$)





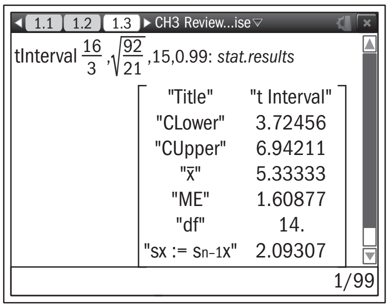
Since the p-value $0.46297 > 0.01$ we have no sufficient evidence to reject the null hypothesis and we conclude that there is no difference in measurement of the two types of biochemical analyzers.

3 a $\bar{x} = \dfrac{\sum_{i=1}^{15} x_i}{15} = \dfrac{80}{15} = \dfrac{16}{3} = 5.33$

$s^2 = \dfrac{\sum_{i=1}^{15} x_i^2 - 15 \times \left(\dfrac{16}{3}\right)^2}{14} = \dfrac{488 - \dfrac{1280}{3}}{14}$

$= \dfrac{184}{42} = \dfrac{92}{21} = 4.38$

b $s = \sqrt{s^2} = \sqrt{\dfrac{92}{21}} = 2.09$



99% CI [3.72, 6.94]

c In 99% of the cases the mean value of a sample of 15 observations taken from the population will fall within the confidence interval [3.72, 6.94].

4 a $\bar{x} = \dfrac{47.2 + 55.2}{2} = 51.2$

b $\sigma^2 = 25 \Rightarrow \sigma = 5 \Rightarrow z_{\frac{\alpha}{2}} \times \dfrac{5}{\sqrt{6}}$

$= 55.2 - 51.2 \Rightarrow z_{\frac{\alpha}{2}} = \dfrac{4\sqrt{6}}{4}$

$= 1.960 \Rightarrow \dfrac{\alpha}{2} = \Phi(1.960) = 0.975$

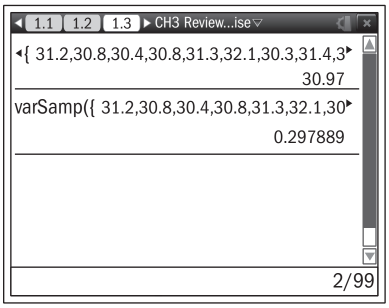
The confidence level of the interval is 95%.

5 a $s = 1000 \text{ m}, v = 120 \text{ km/h} = \dfrac{100}{3} m/s,$

$t = \dfrac{s}{v} \Rightarrow t = \dfrac{1000}{\frac{100}{3}} = 30\,s$

b $\bar{x} = \dfrac{\sum_{i=1}^{10} x_i}{10} = 30.97$

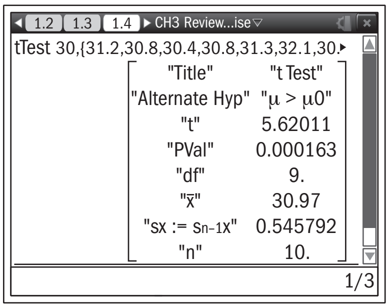
$s^2 = \dfrac{\sum_{i=1}^{10} x_i^2 - 10 \times \left(\dfrac{3097}{100}\right)^2}{9} = \dfrac{2681}{9000} = 0.298$



c H_0: "The average time is 30 s." ($\mu = 30$)

H_1: "The average time is more than 30 s." ($\mu > 30$)

We use the t-test since the standard deviation is unknown.



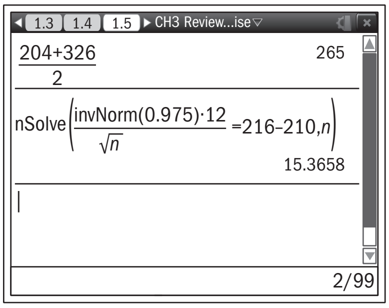
Since the p-value $0.000163 < 0.05$ we reject the null hypothesis and can conclude that the average time is more than 30 seconds. Therefore, the company sets up the speedometers to show a higher speed.

6 a $\bar{x} = \dfrac{204 + 216}{2} = 210$

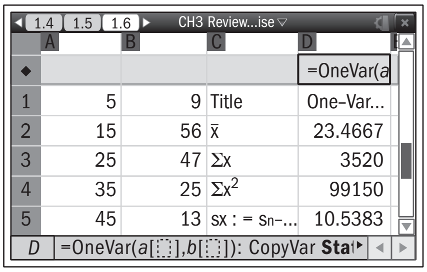
b $X \sim N(210, 144) \Rightarrow \sigma = 12 \Rightarrow 1.960 \times \dfrac{12}{\sqrt{n}}$

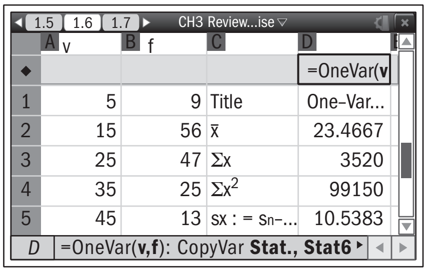
$= 216 - 210 \Rightarrow \sqrt{n} = 3.92 \Rightarrow n = 15.37$

The sample size should be 16.



7 **a** For the speed values we used the midpoints of the intervals.

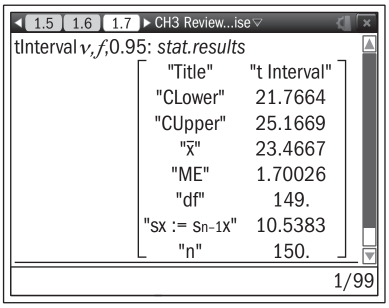




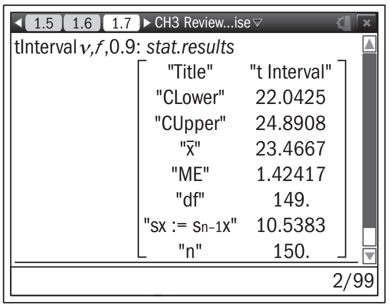
i $= \bar{x} = 23.5$

ii $= s = 10.5$

b **i** = 95% CI [21.8, 25.2]



ii 90% CI [22.0, 24.9]



c We notice that [22.0, 24.9] ⊂ [21.8, 25.2], so a 90% confidence interval is a subset of a 95% confidence interval.

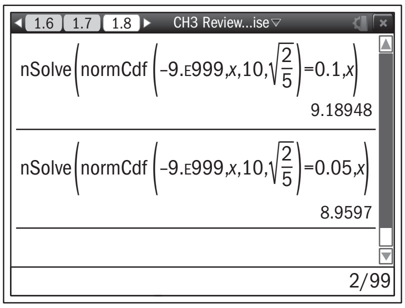
8 $H_0 = \mu = 10$, $H_1: \mu < 10$, using the mean of a sample of size 5.

a $H_0: \mu = 10 \Rightarrow \bar{X} \sim N\left(10, \frac{2}{5}\right)$

i 10% for N (0, 1) is –1.282 so $\dfrac{\bar{x}-10}{\sqrt{\frac{2}{5}}}$

$= -1.282 \Rightarrow \bar{x} = 10 - 1.282 \times \sqrt{\frac{2}{5}} = 9.19$

The critical region is the interval]−∞, 9.19[



ii 5% for N (0, 1) is –1.645 so $\dfrac{\bar{x}-10}{\sqrt{\frac{2}{5}}}$

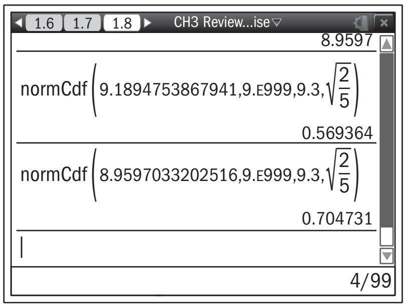
$= -1.645 \Rightarrow \bar{x} = 10 - 1.645 \times \sqrt{\frac{2}{5}} = 8.96$

The critical region is the interval]−∞, 8.96[

b $H_1: \mu = 9.3 \Rightarrow \bar{X} \sim N\left(9.3, \frac{2}{5}\right)$

i $\beta = P(\bar{X} > 9.19) = 0.569$

ii $\beta = P(\bar{X} > 8.96) = 0.705$



c When the probability of a Type I error decreases from 10% to 5%, the probability of a Type II error increase from 0.569 to 0.705.

4 Statistical modeling

Skills Check

1 **a** $2E(Z) - 3E(Y) + 2E(X)$

b $4Var(Z) - 9Var(Y) + 4Var(X)$

c $E(X)E(Y)E(Z)$

2 **a** $x = 43.5$, $y = -31.75$,

$Var(X) = 46.25$, $Var(Y) = 98.69$

b

	A	B ycoord	C	D	E	F	G	H
=					=TwoVar('			
1	33	-21		Title	Two-va...			
2	49	-44		x̄	43.5			
3	50	-23		Σx	174.			
4	42	-39		Σx²	7754.			
5				sx := sn-...	7.85281			
6				σx := σn...	6.80074			
7				n	4.			
8				ȳ	-31.75			
9				Σy	-127.			
10				Σy²	4427.			
11				Sy := Sn-...	11.471			
12				σy := σn...	9.93416			
13				Σxy	-5637.			

E1 ="Two-Variable Statistics"

3 **a** 0.382 **b** 0.951

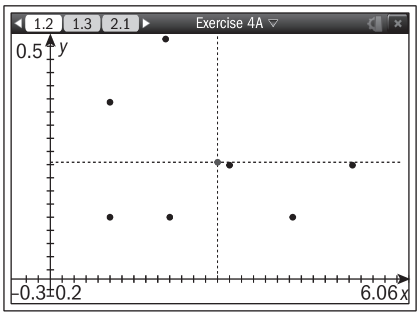
c 0.938 **d** 0.732

1.1 1.2 1.3 CH4 Skills check

tCdf(-0.4,0.8,2)	0.382266
tCdf(-∞,1.83,10)	0.951412
tCdf(-1.75,∞,7)	0.938203
tCdf(-1.14,1.14,20)	0.732245

Exercise 4A

1

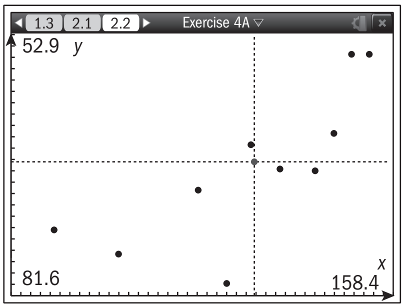


	A xcoord	B ycoord	C	D	E	F	G	H
=					=TwoVar('			=LinRegM
1	4	1		Title	Two-va...		Title	=Linear R..
2	1	1		x̄	2.8		RegEqn	m*x+b
3	3	2		Σx	28.		m	-0.3409..
4	2	4		Σx²	96.		b	2.95455
5	1	3		sx := sn-...	1.39841		r²	0.146104
6	2	4		σx := σn...	1.32665		r	-0.3822..
7	5	2		n	10.		Resid	{-0.5909..
8	4	1		ȳ	2.			
9	4	1		Σy	20.			
10	2	1		Σy²	54.			
11				Sy := Sn-...	1.24722			
12				σy := σn...	1.18322			
13				Σxy	50.			

H1 ="Linear Regression (mx+b)"

Since $r = -0.382$, there is a weak negative correlation between the two random variables.

2

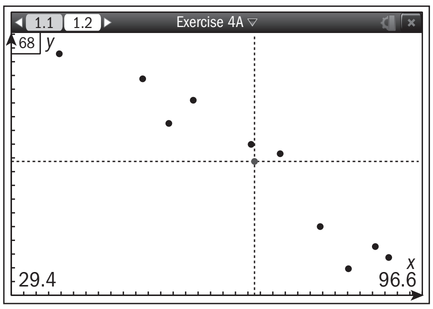


	A xcoord	B ycoord	C	D	E	F	G	H
=					=TwoVar('			=LinRegM
1	100	33		Title	Two-Va...		Title	Linear R...
2	115	38		x̄	126.9		RegEqn	m*x+b
3	140	40		Σx	1269.		m	0.245495
4	149	51		Σx²	165059.		b	9.44674
5	88	36		sx := sn-...	21.1421		r²	0.630724
6	132	40		σx := σn...	20.0572		r	0.794182
7	152	51		n	10.		Resid	{-0.9961..
8	144	43		ȳ	40.6			
9	121	32		Σy	406.			
10	128	42		Σy²	16868			
11				Sy := Sn-...	6.53537			
12				σy := σn...	6.2			
13				Σxy	52509.			

B11

Since $r = 0.794$, there is a strong positive correlation between the two random variables.

3



	A xcoord	B ycoord	C	D	E	F	G	H
=					=TwoVar('			=LinRegM
1	55	58		Title	Two-Va...		Title	Linear R...
2	35	65		$\bar{x}$	66.7		RegEqn	m*x+b
3	66	52		Σx	667.		m	-0.5559...
4	82	35		Σx²	47645.		b	86.1833
5	91	36		sx := sn-...	18.7264		r²	0.939034
6	79	42		σx := σn...	17.7654		r	-0.9690...
7	48	60		n	10.		Resid	{2.39513..
8	52	55		$\bar{y}$	49.1			
9	71	50		Σy	491.			
10	88	38		Σy²	25147.			
11				Sy := Sn-...	10.744			
12				σy := σn...	10.1926			
				Σxy	30995.			

A11

Since $r = -0.970$, there is a strong negative correlation between the two random variables.

Exercise 4B

1 $\text{Cov}(X, Y) = \text{E}(XY) - \mu_x\mu_y$

$= \text{E}(XY) - \mu_y\mu_x$

$= \text{Cov}(Y, X)$

2 $\text{Cov}(X, X) = \text{E}(XX) - \mu_x\mu_y$

$= \text{E}(X^2) - \mu_x^2$

$= \text{Var}(X)$

3 $\text{Cov}(aX, Y) = \text{E}(aXY) - \text{E}(aX)\text{E}(Y)$

$= a\text{E}(XY) - a\text{E}(X)\text{E}(Y)$

$= a[\text{E}(XY) - (\mu_x\mu_y)]$

$= a\text{Cov}(X, Y)$

4 $\text{Cov}(X, bY) = \text{E}(X(bY)) - \text{E}(X)\text{E}(bY)$

$= b\text{E}(XY) - \text{E}(X)\text{E}(bY)$

$= b\text{E}(XY) - b\text{E}(X)\text{E}(Y)$

$= b[\text{E}(XY) - (\mu_x\mu_y)]$

$= b\text{Cov}(X, Y)$

5 $\text{Cov}(X_1 + X_2, Y) = \text{E}\left[(X_1 + X_2)Y\right] - \text{E}(X_1 + X_2)\text{E}(Y)$

$= \text{E}\left[X_1Y + X_2Y\right]$

$-\left[\text{E}(X_1) + \text{E}(X_2)\right]\text{E}(Y)$

$= \left[\text{E}(X_1Y) - \text{E}(X_1)\text{E}(Y)\right]$

$+ \left[\text{E}(X_2Y) - \text{E}(X_2)\text{E}(Y)\right]$

$= \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$

6 $\text{Cov}(X, Y_1, + Y_2) = \text{E}[(X)(Y_1 + Y_2)] - \text{E}(X)\text{E}(Y_1 + Y_2)$

$= \text{E}[XY_1 + XY_2] - \text{E}(X)[\text{E}(Y_1)$
$+ \text{E}(Y_2)]$

$= [\text{E}(XY_1) - \text{E}(X)\text{E}(Y_1)] + [\text{E}(XY_2)$
$- \text{E}(X)\text{E}(Y_2)]$

$= \text{Cov}(X, Y_1) + \text{Cov}(X, Y_2)$

7 $\text{Var}(X + Y) = \text{E}(X + Y)^2 - [\text{E}(X + Y)]^2$

$= \text{E}(X^2 + 2XY + Y^2) - (\mu_x\mu_y)^2$

$= \text{E}(X^2 + 2XY + Y^2) - \mu_x^2 - 2\,\mu_x\mu_y - \mu_y^2$

$= (\text{E}(X^2) - \mu_y^2) + (\text{E}(Y^2) - \mu_y^2)$
$+ 2(\text{E}(XY) - \mu_x\mu_y)$

$= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

8 Since X and Y are independent variables,

$\text{E}(XY) = \text{E}(X)\text{E}(Y) = \mu_x\mu_y$.

Hence, $\text{Cov}(X, Y) = \text{E}(XY) - \mu_x\mu_y = \mu_x\mu_y - \mu_x\mu_y = 0$.
Hence, $p = 0$.

Exercise 4C

1

	A xcoord	B ycoord	C	D	E	F	G
=				=LinRegtT			
1	42	34	Title	Linear R...			
2	2	18	Alternate...	ß & ρ ≠ ...			
3	26	14	RegEqn	a+b*x			
4	22	20	t	4.05922			
5	15	41	Pval	0.002289			
6	44	44	df	10.			
7	50	51	a	9.76713			
8	38	45	b	0.73773			
9	12	17	s	9.69498			
10	28	27	SESlope...	0.181742			
11	22	28	r²	0.622318			
12	1	1	r	0.788871			
13			Resid	{-6.7517...			

D1 ="Linear Reg t Test"

$\text{H}_0: p = 0$; $\text{H}_1: p \neq 0$

Since $0.00229 < 0.01$, we reject the null hypothesis. There is evidence of significant correlation between the two variables at the 1% level.

2

	A xcoord	B ycoord	C	D	E	F	G
=					=LinRegtT		
1	64	69		Title	Linear R...		
2	66	64		Alternate...	ß & ρ ≠ ...		
3	68	67		RegEqn	a+b*x		
4	69	64		t	0.931872		
5	70	62		Pval	0.394177		
6	72	73		df	5.		
7	74	71		a	35.6		
8				b	0.457143		
9				s	4.10435		
10				SESlope...	0.490564		
11				r²	0.147977		
12				r	0.384678		
13				Resid	{4.14285...		

E1 ="Linear Reg t Test"

$\text{H}_0: p = 0$; $\text{H}_1: p = 0$

Since $0.394 > 0.05$, there is not enough evidence to reject the null hypothesis. There is not evidence of significant correlation between the two variables at the 5% level.

3

	A xcoord	B ycoord	C	D	E	F	G
=					=LinRegtT		
1	67	104		Title	Linear R...		
2	70	116		Alternate...	ß & ρ ≠ ...		
3	70	121		RegEqn	a+b*x		
4	70	126		t	2.62304		
5	71	117		Pval	0.034257		
6	72	113		df	7.		
7	72	124		a	-93.		
8	72	126		b	3.		
9	73	127		s	5.78174		
10				SESlope...	1.14371		
11				r²	0.49569		
12				r	0.704052		
13				Resid	{-4.,-0.9...		

E1 ="Linear Reg t Test"

$H_0: p = 0$; $H_1: p = 0$

Since $0.0343 < 0.1$, we reject the null hypothesis. There is evidence of significant correlation between the two variables at the 10% level.

Exercise 4D

1 a

	A score	B age	C he...	D We...	E	F	G	H	I	J
=						=LinRegM		=LinReg		=LinRegM
1	62	10	145	44.9	Title...	Linear R...	Title...	Linear	Title...	Linear R...
2	94	11	150	42.1	Reg...	m*x+b	Reg...	m*x+b	Reg...	m*x+b
3	81	8	139	30.2	m	0.014779	m	0.1642...	m	-0.2780...
4	62	9	148	41.4	b	8.48321	b	131.601	b	61.3778
5	58	8	130	41.8	r²	0.035188	r²	0.0426...	r²	0.298465
6	86	10	150	38.4	r	0.187584	r	0.2064	r	-0.54632
7	59	11	154	54.1	Res...	{0.60049...	Res...	{3.216...	Res...	{0.75945...
8	72	9	140	38.6						
9	54	10	153	52.4						
10	60	9	120	38.6						
11										
12										
13										

E10

Column F shows the regression data for score versus age.
Column H shows the regression data for score versus height.
Column J shows the regression data for score versus weight.
$|r|$ is largest for the random variables score and weight, hence we use W.

b From the GDC, $S = -0.270W + 61.4$

c From the GDC,

	A	B	C	D	E	F	G
=							
1	Title...	Linear R...					
2	Alternate...	ß & ρ ≠ ...					
3	RegEqn	a+b*x					
4	t	-1.84488					
5	PVal	0.102277					
6	df	8.					
7	a	61.3778					
8	b	-0.2780...					
9	s	6.1981					
10	SESlope...	0.150699					
11	r²	0.298465					
12	r	-0.54632					
13	Resid	{0.75945..					

C13

Since $p = 0.102 > 0.05$ there is insufficient evidence to reject the null hypothesis, i.e. there is not evidence of significant correlation between the score and the weight at the 5% level.

2

	A body	B heart	C	D	E	F	G
=					=LinRegM		
1	30	136		Title	Linear R...		
2	37	156		RegEqn	m*x+b		
3	38	150		m	1.90885		
4	32	140		b	82.763		
5	36	155		r²	0.57174		
6	32	157		r	0.756135		
7	33	143		Resid	{-4.0286...		
8	38	160					
9	44	170					
10	38	144					
11							
12							
13							

F1

$r = 0.756$, which indicates a strong correlation between the two variables. Hence we find the line of regression, $y = 1.91x + 82.8$, and when $x = 35\,g$, $y = 150$ mg.

3 a $H_0: p = 0$; $H_1: p = 0$

b

	A x	B y	C	D	E	F	G
=					=LinRegM		=LinRegT
1	120	80		Title	Linear R...	Title	Linear R...
2	125	90		RegEqn	m*x+b	Alternate...	ß & p ≠ ...
3	130	92		m	0.62303	RegEqn	a+b*x
4	135	98		b	10.8182	t	10.0189
5	140	100		r²	0.926185	Pval	0.000008
6	145	103		r	0.962385	df	8.
7	150	105		Resid	{-5.5818...	a	10.8182
8	155	108				b	0.62303
9	160	110				s	2.82414
10	165	110				SESlope...	0.062185
11						r²	0.926185
12						r	0.962385
13						Resid	{-5.5818...

G1

$r = 0.962$; $p = 8 \times 10^{-6}$

c There is very strong evidence to indicate a positive association between the random variables X and Y.

d Since r is close to 1, we can find the regression line, $y = 0.623x + 10.8$, and when $x = 138$ mm, $y = 97$ mm.

4 $$m = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sum y^2 - n\bar{y}^2} = \frac{8390 - 5\left(\frac{182}{5}\right)\left(\frac{200}{5}\right)}{9850 - 5\left(\frac{200}{5}\right)^2} = 0.6$$

$x - \bar{x} = 0.6(y - \bar{y}) \Rightarrow x = 0.6y + 12.4$; $x = 66.4$ g

5 For the gradient of the Y on X line of regression,

$$m = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sum x^2 - n\bar{x}^2} = \frac{580 - 10\left(\frac{30}{10}\right)\left(\frac{86}{10}\right)}{220 - 10\left(\frac{30}{10}\right)^2} = 2.48, \text{ hence}$$

$y - \bar{y} = 2.48(x - \bar{x}) \Rightarrow y = 2.48x + 1.17$

For the gradient of the X on Y line of regression,

$$m = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sum y^2 - n\bar{y}^2} = \frac{580 - 10\left(\frac{30}{10}\right)\left(\frac{86}{10}\right)}{1588 - 10\left(\frac{86}{10}\right)^2} = 0.380, \text{ hence}$$

$x - \bar{x} = 0.380(y - \bar{y}) \Rightarrow x = 0.380y - 29.6$

Review exercise

1 a r is the unbiased estimate of ρ, and

$$r = \frac{\sum x_i y_i - n\bar{x}\,\bar{y}}{\sqrt{\left(\sum x_i^2 - n\bar{x}^2\right)\left(\sum y_i^2 - n\bar{y}^2\right)}} = 0.975$$

$$t = r\sqrt{\frac{n-2}{1-r^2}} = 10.7$$

Hence, since $p = .000039 < 0.05$, there is evidence of a strong positive relationship between the two random variables.

b $$m = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sum y^2 - n\bar{y}^2} = \frac{37\,000 - 8\left(\frac{440}{8} \times \frac{606}{8}\right)}{49\,278 - 8\left(\frac{606}{8}\right)^2} = 1.088$$

$x - \bar{x} = 1.088(y - \bar{y}) \Rightarrow x = 1.088y - 27.4;$

$x = 32.4$

2 Since $\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$, the maximum $\text{Cov}(X, Y)$ will occur when $\rho = 1$.

Hence, $\text{Cov}_{max}(X, Y) = \sqrt{17 \times 15} = 10.2$

3 If $p = 0.9$ then $t = invt(0.9, 18) = 1.33039$, and solving $1.33039 = r\sqrt{\frac{18}{1-r^2}}$, gives $r = 0.299$

4 a $$\rho = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}} = \frac{\text{Cov}((X+Y), (X+Z))}{\sqrt{\text{Var}(X+Y)\text{Var}(X+Z)}}$$

$$= \frac{\text{Cov}(XX) + \text{Cov}(XZ) + \text{Cov}(YX) + \text{Cov}(YZ)}{\sqrt{[\text{Var}(X+Y)][\text{Var}(X+Z)]}}$$

$$= \frac{\text{Var}(X) + 0}{\sqrt{2\sigma^2}\sqrt{2\sigma^2}} = \frac{\sigma^2}{2\sigma^2} = \frac{1}{2}$$

Hence it cannot be said that U and V are independent.

b $$\rho = \frac{\text{Cov}(V, W)}{\sqrt{\text{Var}(V)\text{Var}(W)}} = \frac{\text{Cov}((X+Y), (X-Y))}{\sqrt{\text{Var}(X+Y)\text{Var}(X-Y)}}$$

$$= \frac{\text{Cov}(XX) - \text{Cov}(XY) + \text{Cov}(YX) - \text{Cov}(YY)}{\sqrt{[\text{Var}(X+Y)][\text{Var}(X-Y)]}}$$

$$= \frac{\text{Var}(X) - \text{Var}(Y)}{\sqrt{[\text{Var}(X+Y)][\text{Var}(X-Y)]}} = 0$$

We know that if V and W are independent then $\rho = 0$, but the converse does not necessarily hold.

5 a

	A xcord	B ycord	C	D	E	F	G
=				=LinRegT			
1	44.5	41.2	Title	Linear R...			
2	10.3	11.1	Alternate...	ß & ρ ≠ ...			
3	20.1	18.7	RegEqn	a+b*x			
4	55	52.3	t	20.9954			
5	39.6	41.2	PVal	2.78035...			
6	24.1	26.5	df	8.			
7	31.2	29.3	a	2.59473			
8	9.5	11.2	b	0.904534			
9	22.3	25.1	s	1.9017			
10	35.1	33.2	SESlope...	0.043083			
11			r²	0.982175			
12			r	0.991047			
13			Resid	{-1.6465...			

D5 =2.7803542327572E-8

From the GDC, $r = 0.991$; $p = 2.78 \times 10^{-8}$

b The p-value suggests a strong relationship between the two random variables, hence it makes sense to find the equation of the regression line: $y = 0.905x + 2.59$

c $y = 0.905(19.8) + 2.59 = 20.5$

d $p = 0.00620 < 0.01$, hence there is a significant correlation between the random variables.

6 a The gradient of the Y on X line, $y = a + bx$, is $\frac{\text{Cov}(X, Y)}{\text{Var}(X)}$, and the gradient of the X on Y line, $x = c + dy$, is $\frac{\text{Cov}(X, Y)}{\text{Var}(Y)}$. Since $r = 0$, $\text{Cov}(X, Y) = 0$, hence $y = a$ and $x = b$, and the two lines are perpendicular.

b For $y = a + bx$, $m = b$, and for $x = c + dy$, $m = \frac{1}{d}$.

$r = \pm 1 \Rightarrow r^2 = 1$. Since $r^2 = bd \Rightarrow bd = 1$, $b = \frac{1}{d}$.

Hence, the two lines have the same gradient. Since both regression lines must go through $(\bar{x}, \bar{y})$, the lines are identical.

c For the regression lines $y = a + bx$ and $x = c + dy$,

$b = \frac{\text{Cov}(X, Y)}{\text{Var}(X)}$ and $d = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}$. Hence

$bd = \left[\frac{\text{Cov}(X, Y)}{\text{Var}(X)\text{Var}(Y)}\right]^2 = r^2$. Since b and d are either both positive or both negative, $r = +\sqrt{bd}$ if both are positive, and $r = -\sqrt{bd}$ if both are negative.

d Since b and d are both positive,

$r = +\sqrt{bd} = \sqrt{0.19 \times 0.77} = 0.382$